

# CONFORMAL INVARIANCE OF BOUNDARY TOUCHING LOOPS OF FK ISING MODEL

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**ABSTRACT.** In this article we show the convergence of a loop ensemble of interfaces in the FK Ising model at criticality, as the lattice mesh tends to zero, to a unique conformally invariant scaling limit. The discrete loop ensemble is described by a canonical tree glued from the interfaces, which then is shown to converge to a tree of branching SLEs. The loop ensemble contains unboundedly many loops and hence our result describes the joint law of infinitely many loops in terms of SLE type processes, and the result gives the full scaling limit of the FK Ising model in the sense of random geometry of the interfaces.

Some other results in this article are convergence of the exploration process of the loop ensemble (or the branch of the exploration tree) to  $\text{SLE}(\kappa, \kappa - 6)$ ,  $\kappa = 16/3$ , and convergence of a generalization of this process for 4 marked points to  $\text{SLE}[\kappa, Z]$ ,  $\kappa = 16/3$ , where  $Z$  refers to a partition function. The latter SLE process is a process that can't be written as a  $\text{SLE}(\kappa, \rho_1, \rho_2, \dots)$  process, which are the most commonly considered generalizations of SLEs.

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## 1. INTRODUCTION

**1.1. The setup informally.** The *Ising model* is one of the most studied lattice models in statistical physics. The Ising model (and Potts models generalizing it) have percolation-type representations called *Fortuin–Kasteleyn random cluster models* (FK model). Given a graph the Ising model assigns probabilities to configurations of  $\pm 1$  spins on the vertices of the graph and the random cluster model assigns probabilities to configurations of open/closed edges. FK model is obtained from the percolation model (giving weight  $p$  to each open edge and  $1 - p$  to each closed edge) by weighting by a factor  $q$  per each connected cluster of open edges. We will consider the FK Ising model, i.e., FK model whose parameter value  $q$  corresponds to the Ising model, on a square lattice. The lattice we are considering is the usual square lattice  $\mathbb{Z}^2$ , for convenience rotated by  $45^\circ$ . In Figure 1(a) it is the lattice formed by the centers of the black squares.

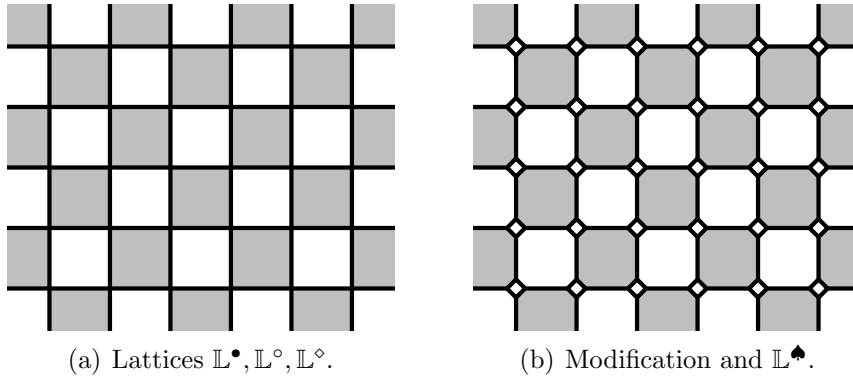


FIGURE 1. The square lattices we are considering are  $\mathbb{L}^\bullet$  formed by the centers of the black squares,  $\mathbb{L}^\circ$  formed by the centers of the white squares and  $\mathbb{L}^\diamond$  formed by the corners of the black and white squares. We will also consider the square–octagon lattice  $\mathbb{L}^\spadesuit$  which we see as a modification of  $\mathbb{L}^\diamond$ .

We will consider a bounded simply connected subgraph of the square lattice. We will make this definition clearer later. An FK configuration in the graph is illustrated in 2(a) and its dual configuration in 2(b). The dual configuration is defined on the

dual lattice formed by the white squares in Figure 1(a) with the rule that exactly one of any edge and its dual edge (the one crossing the edge in  $90^\circ$  angle) is present in the configuration or in the dual configuration. The so called *loop representation* of the FK model is defined on so called *medial lattice* which is in Figure 1(a) formed by the corners of the squares, by taking the inner and outer boundaries of all the connected components in a configuration of edges. The loop representation can be seen as a dense collection of *simple* loops when we resolve the possible double vertices by the modification illustrated in Figure 1(b) or Figure 2.

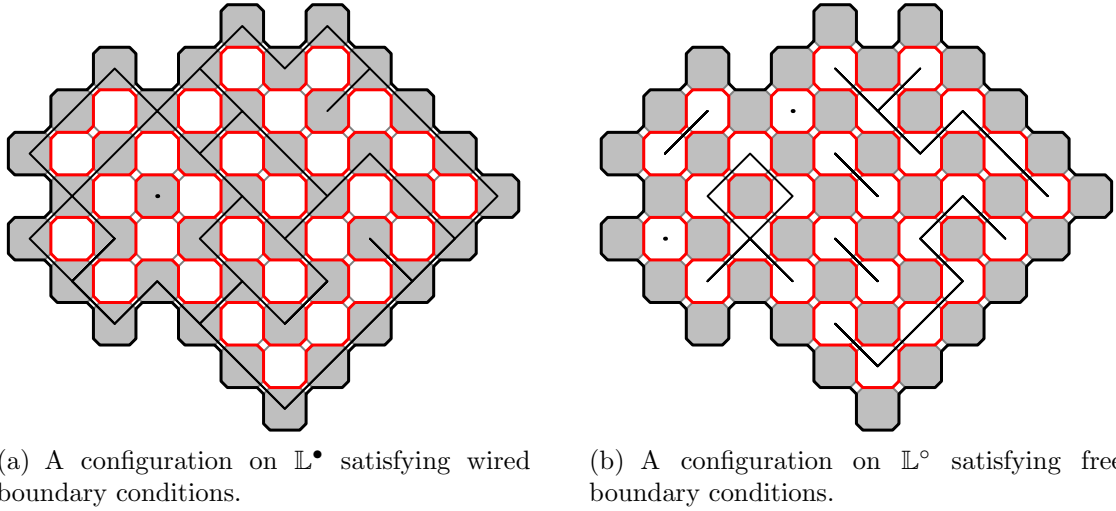


FIGURE 2. FK configurations with wired and free boundary. Notice that these two configurations are dual to each other and thus have the same loop representations.

The duality of the FK configurations described above gives a mapping between the set of configurations with wired and one with free boundary conditions, defined on the graph and on its dual, respectively. It defines also an involution among the probability laws of FK model. One can check that the FK model parameter values  $p$  and  $q$  get mapped to values  $p^*$  and  $q^* = q$  where  $p^*$  is given by

$$\frac{pp^*}{(1-p)(1-p^*)} = q. \quad (1)$$

In this article we will consider critical point of the model which happens to be the self-dual value of  $p$ , that is,  $p = p_c$  when  $p = p^*$ . For the FK Ising model, the parameter  $q = 2$  and the *critical parameter*  $p_c = \frac{\sqrt{2}}{1+\sqrt{2}}$ . By the fact that it is the self-dual point we see that in Figures 2(a) and 2(b) the FK models have the same parameter values. The only difference is in boundary conditions. The resulting loop configurations have the same law for the critical parameter on both setups, either wired boundary conditions on the primal graph or free boundary conditions on the dual graph.

The correspondence of boundary conditions is slightly more complicated for other boundary conditions than the wired and free ones.

**1.1.1. The role of the critical parameter.** The parameter is chosen to be critical for a couple of reasons. First of all it is expected that only for this value of the parameter the scaling limit will be non-trivial. For the other values, we get either zero or one macroscopic loop and all the other loops will be microscopic and vanishing in the

limit. The only macroscopic loop will be rather uninteresting as it will follow the boundary. In contrast, for the critical parameter the scaling limit will consist of (countably) infinite number of loops as we will be showing.

The second reason for selecting this value of the parameter is more technical. For that value, the observable we are defining in Section 4.1 is going to satisfy a relation which we can interpret as a discrete version of the Cauchy–Riemann equations. This makes it possible to pass to the limit and recover in the limit a holomorphic function solving a boundary value problem.

The third reason for the choice of the critical parameter is that at criticality we expect that the loop collection will have a conformal symmetry. This is already suggested by the existence of the holomorphic observable.

## 1.2. The exploration tree and the main results.

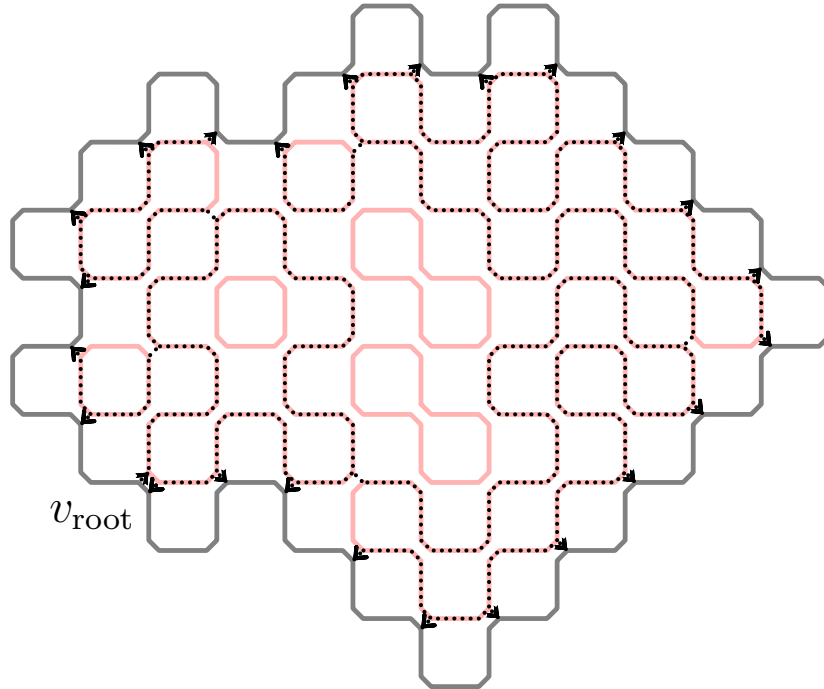


FIGURE 3. The loop collection and the exploration tree: solid pink and solid pink on the background of dotted lines indicate the loops. The dotted black lines and the solid black lines form the tree. The solid black lines are the jumps from one loop to another.

1.2.1. *The exploration tree.* Figure 3 illustrates the construction of the *exploration tree* of a loop configuration. The (chordal) exploration tree connects the fixed root vertex to any other boundary point. The construction of the branch from the root to a fixed target vertex is the following:

- (1) To initialize the process cut open the loop next to the root vertex and start the process on that loop.
- (2) Explore the current loop in clockwise direction until you reach the target point or you arrive to a point at which it is clear that it is impossible to reach the target point along the current loop (meaning that we disconnected ourselves from the target).
- (3) If you reached the target, stop and return the path which is concatenated from the subpaths of the loops that were explored during the process in the order that they were explored.

- (4) If you haven't reach the target, cut open the loop next to your current position and jump to that loop and go to Step (2).

The construction depends on the direction which was chosen above to be clockwise. We could also use, say, independent coin flips to decide whether to follow each loop in clockwise or counterclockwise direction. However we will mostly use the above construction which suits well our purposes. After all, the main goal is to show convergence of the loop collection, and the above construction agrees well with our observable.

**1.2.2. The main results.** The main theorem is the following result. Let's call a discrete domain *admissible* if it is simply connected and the boundary consists of black squares as in Figure 2(a).

**Theorem 1.1.** *Let  $\Omega_{\delta_n}$  be a sequence of admissible domains that converges to a domain  $\Omega$  in Carathéodory sense with respect to some fixed  $w_0 \in \Omega$  and let  $\Theta_{\partial, \delta_n}$  be the random collection of loops which is the collection of all loops of the loop representation of the FK Ising model on  $\Omega_{\delta_n}$  that intersect the boundary. Let  $\phi_n$  be a conformal map that sends  $\Omega_n$  onto the unit disc and  $w_0$  to 0. Then as  $n \rightarrow \infty$ , the sequence of random loop collections  $\phi_n(\Theta_{\partial, \delta_n})$  converges weakly to a limit  $\Theta$  whose law is independent of the choices of  $\Omega, \delta_n, \Omega_{\delta_n}, w_0$  and  $\phi_n$ .*

*Moreover the law of  $\Theta$  is given by the image of the  $SLE(\kappa, \kappa - 6)$  exploration tree with  $\kappa = 16/3$  under a tree-to-loops mapping which inverts the construction in Section 1.2.1 and is explained in more details in Section 2.3.*

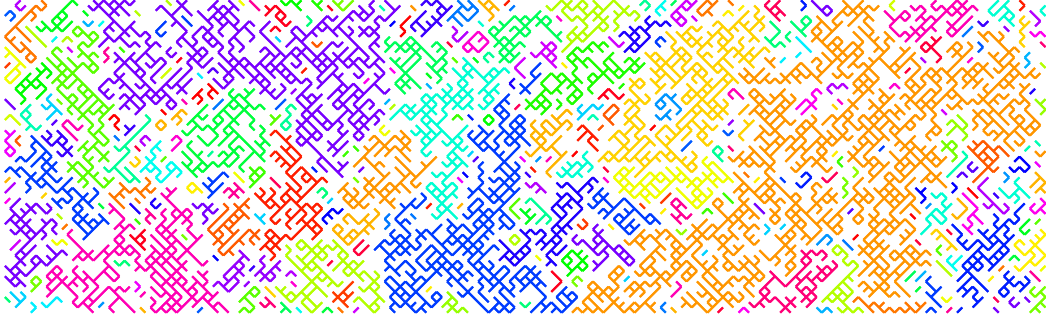
**Remark 1.2.** In this article we will assume that  $\Omega$  is smooth and that each approximation has boundary that is piece-wise approximatively linear in a certain sense. We will exclude the cases where the boundary forms long fjords to have better estimates for the harmonic measure. The general case follows from the sequel [12] of this article on the *radial* exploration tree of the FK Ising model. These restrictions are technical and they are not needed, for instance, for the convergence in the so called 4-point case (Section 4.4.1 and other arguments leading to the convergence of the interface to  $SLE[\kappa, Z]$  in Section 5.5).

The present article aims to provide clear arguments for the regularity properties of trees, derivation of the martingale observables and the corresponding martingale characterization in a boundary touching setting. In principle, one should be able to deduce the complete picture by repeatedly iterating this construction inside the resulting holes appearing after removing the boundary touching loops, the main difficult point being the fractal boundary. Instead, in the sequel [12] we build the complete tree towards interior points, thus not having to deal with fractal boundaries. This requires working with a more complicated observable, and the proof in the current article better explains what follows in the sequel.

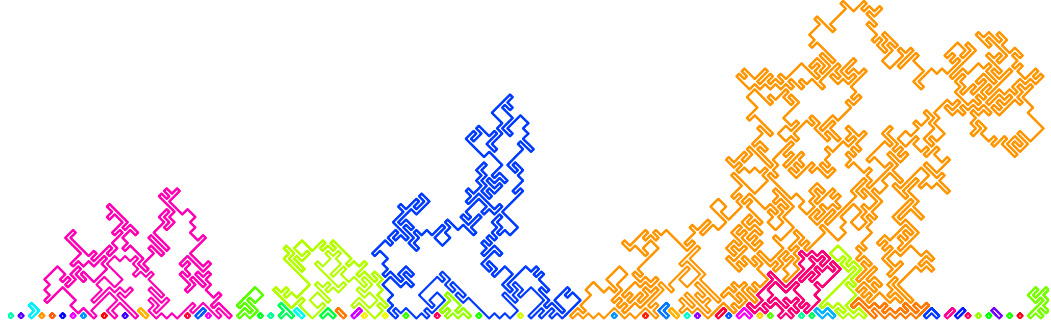
Our result in the 4-point setting is interesting in its own right. In a follow-up paper [13], we use it to show that the interface conditioned on an internal arc pattern converges towards so-called hypergeometric SLE.

A sample of FK Ising branch is illustrated in Figure 4.

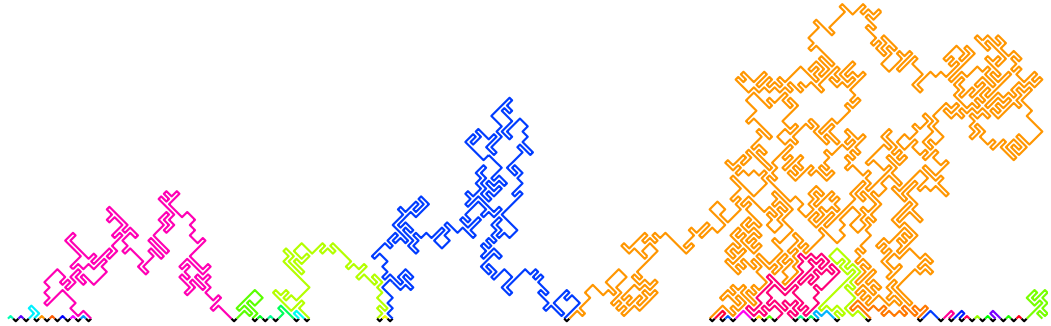
**1.3. The organization of the article.** We will give further definitions in Section 2. In Section 3 we explore the regularity and tightness properties of the loop configurations and the exploration trees based on crossing estimates. This gives a priori knowledge needed in the main argument. In Section 4 we define the holomorphic observable and show its convergence. In Section 5 we combine these tools



(a) A sample of FK Ising model with free boundary conditions.



(b) The boundary touching loops of the loop representation which intersect the bottom of the rectangle.



(c) The exploration process between the lower left and right corners. Notice that the process uses only some loops touching the bottom arc and it uses only the loop arcs which are at top as seen from the lower right corner.

FIGURE 4. A sample configuration of the critical FK Ising model on a rectangular domain with free boundary conditions. Components with only one vertex are not shown. The colors distinguish different components. Notice that in this particular sample, the large orange loop happens to come fairly close to the boundary without disconnecting the small loops to its right. Thus the exploration path turns and explores those boundary touching loops inside the “fjord.”

and extract information from the observable so that we can characterize the scaling limit and prove the main theorem in Section 6.

## 2. THE SETUP AND STATEMENT OF THE MAIN RESULTS

**2.1. Graph theoretical notations and setup.** In this article the lattice  $\mathbb{L}^\bullet$  is the square lattice  $\mathbb{Z}^2$  rotated by  $\pi/4$ ,  $\mathbb{L}^\circ$  is its dual lattice, which itself is also a square lattice and  $\mathbb{L}^\diamond$  is their (common) medial lattice. More specifically, we define three

lattices  $G = (V(G), E(G))$ , where  $G = \mathbb{L}^\bullet, \mathbb{L}^\circ, \mathbb{L}^\diamond$ , as

$$V(\mathbb{L}^\bullet) = \{(i, j) \in \mathbb{Z}^2 : i + j \text{ even}\}, \quad E(\mathbb{L}^\bullet) = \{\{v, w\} \subset V(\mathbb{L}^\bullet) : |v - w| = \sqrt{2}\}, \quad (2)$$

$$V(\mathbb{L}^\circ) = \{(i, j) \in \mathbb{Z}^2 : i + j \text{ odd}\}, \quad E(\mathbb{L}^\circ) = \{\{v, w\} \subset V(\mathbb{L}^\circ) : |v - w| = \sqrt{2}\}, \quad (3)$$

$$V(\mathbb{L}^\diamond) = (1/2 + \mathbb{Z})^2, \quad E(\mathbb{L}^\diamond) = \{\{v, w\} \subset V(\mathbb{L}^\diamond) : |v - w| = 1\} \quad (4)$$

Notice that sites of  $\mathbb{L}^\diamond$  are the midpoints of the edges of  $\mathbb{L}^\bullet$  and  $\mathbb{L}^\circ$ .

We call the vertices and edges of  $V(\mathbb{L}^\bullet)$  *black* and the vertices and edges of  $V(\mathbb{L}^\circ)$  *white*. Correspondingly the faces of  $\mathbb{L}^\diamond$  are colored black and white depending whether the center of that face belongs to  $V(\mathbb{L}^\bullet)$  or  $V(\mathbb{L}^\circ)$ .

The directed version  $\mathbb{L}^\diamond_\rightarrow$  is defined by setting  $V(\mathbb{L}^\diamond_\rightarrow) = V(\mathbb{L}^\diamond)$  and orienting the edges around any black face in the counter-clockwise direction.

The modified medial lattice  $\mathbb{L}^\spadesuit$ , which is a square-octagon lattice, is obtained from  $\mathbb{L}^\diamond$  by replacing each site by a small square. See Figure 1. The oriented lattice  $\mathbb{L}^\spadesuit_\rightarrow$  is obtained from  $\mathbb{L}^\spadesuit$  by orienting the edges around black and white octagonal faces in counter-clockwise and clockwise directions, respectively.

**Definition 2.1.** A simply connected, non-empty, bounded domain  $\Omega$  is said to be a *wired  $\mathbb{L}^\spadesuit_\rightarrow$ -domain* (or *admissible domain*) if  $\partial\Omega$  oriented in counter-clockwise direction is a path in  $\mathbb{L}^\spadesuit_\rightarrow$ .

See Figure 5 for an example of such a domain. The wired  $\mathbb{L}^\spadesuit_\rightarrow$ -domains are in one to one correspondence with non-empty finite subgraphs of  $\mathbb{L}^\bullet$  which are simply connected, i.e., they are graphs who have an unique unbounded face and the rest of the faces are squares.

**2.2. FK Ising model: notations and setup for the full scaling limit.** Let  $G$  be a simply connected subgraph of the square lattice  $\mathbb{L}^\bullet$ . Consider the random cluster measure  $\mu = \mu_{p,q}^1$  of  $G$  with all *wired boundary conditions* in the special case of the critical FK Ising model, that is, when  $q = 2$  and  $p = \sqrt{2}/(1 + \sqrt{2})$ . Its dual model is again a critical FK Ising model, now with free boundary conditions on the dual graph  $G^\circ$  of  $G$  which is a (simply connected) subgraph of  $\mathbb{L}^\circ$ . They have common loop representation on the corresponding subgraph  $G^\spadesuit$  of the modified medial lattice  $\mathbb{L}^\spadesuit$ .

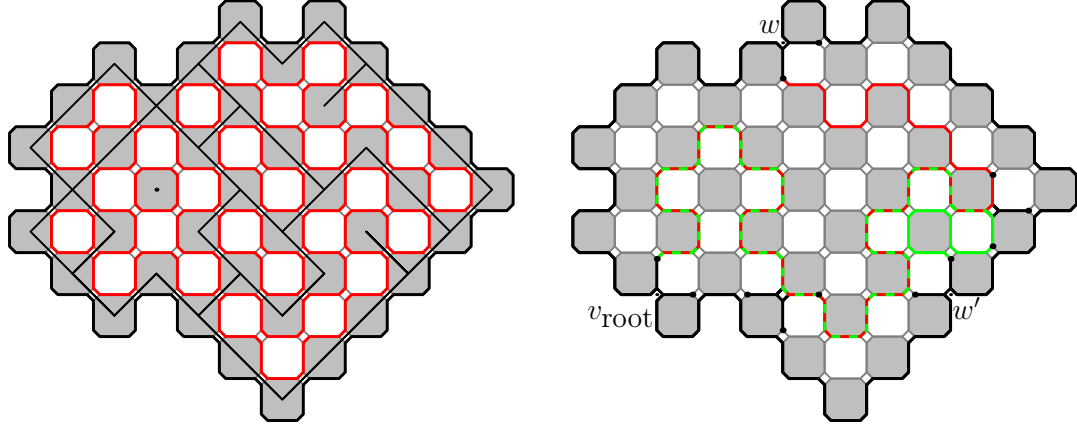
Let's call a collection of loops  $\mathcal{L} = (L_j)_{j=1, \dots, N}$  on  $G^\spadesuit$  *dense collection of non-intersecting loops* (DCNIL) if

- each  $L_j \subset G^\spadesuit$  is a simple loop
- $L_j$  and  $L_k$  are vertex-disjoint when  $j \neq k$
- for every edge  $e \in E^\diamond$  there is a loop  $L_j$  that visits  $e$ . Here we use that  $E^\diamond$  is naturally a subset of  $E^\spadesuit$ .

We consider loop collections only modulo permutations, that is, two objects are equal if they are permutations of each other. Let the collection of all the loops in the loop representation be  $\Theta = (\theta_j)_{j=1, \dots, N}$ . Then DCNIL is exactly the support of  $\Theta$  and for any DCNIL collection  $\mathcal{C}$  of loops

$$\mu(\Theta = \mathcal{L}) = \frac{1}{Z} (\sqrt{2})^{\# \text{ of loops in } \mathcal{L}} \quad (5)$$

where  $Z$  is a normalizing constant.



(a) A sample of random cluster model and the corresponding loop configuration.

(b) Two branches of exploration tree from  $v_{\text{root}} \in V_{\partial}$  to  $w, w' \in V_{\partial}$ . Note the target independence of the process and that the branching takes place on the vertices of  $V_{\partial,1}$ .

FIGURE 5. The construction of the exploration tree of the random cluster model.

We denote the “external” boundary of the domain by  $\partial G^{\blacklozenge}$  and the “internal” boundary, which is the outermost (simple) loop can be drawn in  $G^{\blacklozenge}$ , as  $\partial_1 G^{\blacklozenge}$ , that is,  $\partial G^{\blacklozenge}$  and  $\partial_1 G^{\blacklozenge}$  are as close as possible and the layer of wired edges of  $G^{\bullet}$  lies between them.

Define the collection of *boundary touching loops*,  $\Theta_{\partial} \subset \Theta$ , to be simply the set of loops which intersect the boundary  $\partial_1 G^{\blacklozenge}$ .

Now take a bounded simply connected domain  $\Omega$  in the plane. And take a sequence  $\delta_n \searrow 0$  as  $n \rightarrow \infty$  and a sequence of simply connected graphs  $G_{\delta_n}^{\bullet} \subset \delta_n \mathbb{L}^{\bullet}$  which approximate  $\Omega$  in the sense that, if we denote by  $\Omega_{\delta_n}$  the bounded component of  $\mathbb{C} \setminus \partial G_{\delta_n}^{\blacklozenge}$ , then  $\Omega_{\delta_n}$  converges in Carathéodory convergence to  $\Omega$  (with respect to any interior point of  $\Omega$ ). Fix any  $w_0 \in \Omega$  and let  $\phi_{\delta_n} : \Omega_{\delta_n} \rightarrow \mathbb{D}$  be conformal transformations normalized in the usual way using  $w_0$ , that is,

$$\phi_{\delta_n}(w_0) = 0, \quad \phi'_{\delta_n}(w_0) > 0.$$

Set  $\tilde{\Theta}_{\partial, \delta_n} = \phi_{\delta_n}(\Theta_{\partial, \delta_n})$ .

Let us repeat here the first half of Theorem 1.1.

**Theorem 1.1 (a)** (Conformal invariance of  $\Theta_{\partial}$ ). *As  $n \rightarrow \infty$ ,  $\tilde{\Theta}_{\partial, \delta_n}$  converges weakly to a random collection  $\Theta$  of loops in  $\mathbb{D}$ . The law of  $\Theta$  is independent of  $\Omega, \delta_n, \Omega_{\delta_n}$  and  $w_0$ .*

**2.3. The exploration tree of FK Ising model.** Remember that  $\partial\Omega$  and  $\partial_1\Omega$  are simple loops on  $\mathbb{L}_{\rightarrow}^{\blacklozenge}$  and that  $\Theta_{\partial}$  was the set of loops in  $\Theta$  that intersected  $\partial_1\Omega$ . Next we will explain the construction of the *exploration tree* of  $\Theta_{\partial}$ . The branches of the tree will be simple paths from a root edge to a directed edge of  $\partial_1\Omega$ . More specifically let  $V_{\text{target}}$  be the vertex set of  $\partial_1\Omega$  and for each  $v \in V_{\text{target}}$ , let  $f_v$  be the edge of  $\partial_1\Omega$  arriving to  $v$ .

We assume that the root vertex  $v_{\text{root}} \in V_{\partial}$  of the exploration is fixed. For any  $w \in V_{\text{target}}$ , we are going to construct a simple path which starts from the inwards pointing edge of  $v_{\text{root}}$  and ends on the edge  $f_w$  of  $w$ , denoting this path by  $T_w = T_{v_{\text{root}}, w}$ . We will call the mapping from the loop collections to the trees the “*loops-to-tree map*.”



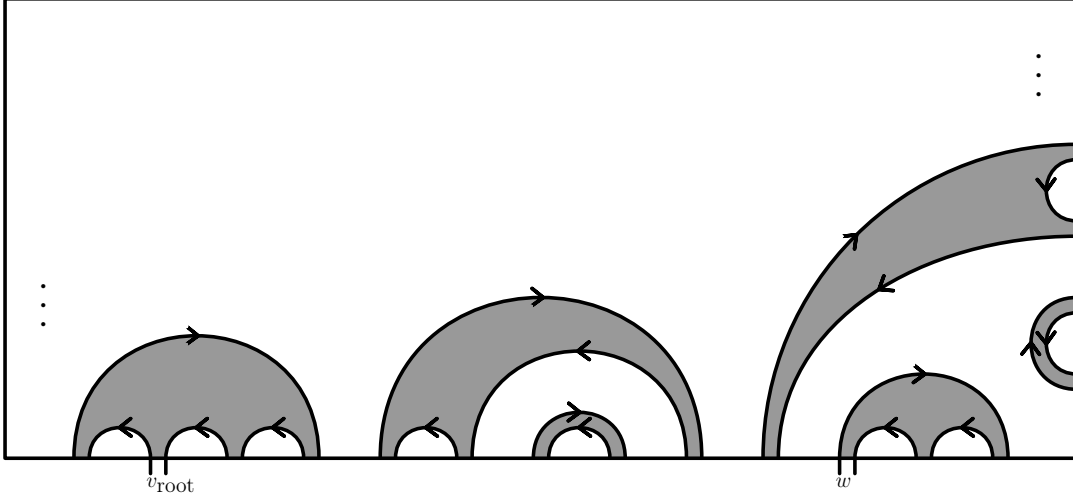


FIGURE 6. A schematic picture of the boundary touching loops. The interiors of the loops are shaded to make it easier to distinguish the loops and the arrows indicate the clockwise orientation of the loops.

Consider a loop collection  $\mathcal{L} = (L_j)_{j \in J}$  where  $J$  is some finite, abstract index set. The index  $j \in J$  shouldn't be confused with the concrete subset  $L_1, \dots, L_n$  we are choosing below. Assume that  $\mathcal{L}$  satisfies the properties of DCNIL. Below we describe the algorithm of Section 1.2.1 for a boundary target point. Note that branching occurs only on the boundary. Namely, the simple path  $T_w$  is constructed in the following way. Here we consider  $L_j$  as a path in  $\mathbb{L}^\diamond$  and lift it to  $\mathbb{L}^\blacklozenge$  when needed.

- (1) Set  $e_0$  to be the inward pointing edge of  $\partial_1\Omega$  at  $v_{\text{root}}$  and  $f_{\text{end}} = f_w$ , that is the edge of  $\partial_1\Omega$  arriving to  $w$ . Denote the set of edges in  $\partial_1\Omega$  that lie between  $f_{\text{end}}$  and  $e_0$ , including  $f_{\text{end}}$ , by  $F_w$ .
- (2) Set  $L_1$  to be the loop going through  $e_0$ . Find the first edge of  $L_1$  after  $e_0$  in the orientation (remember that all the loops are oriented in the clockwise direction) of  $L_1$  lies in  $F_w$ . Call it  $f_1$  and the part of  $L_1$  between  $e_0$  and  $f_1$ , not including  $f_1$ ,  $L_1^T$ .

Notice that  $L_1$  goes through  $f_{\text{end}}$  if and only if  $f_1$  is equal to  $f_{\text{end}}$ . Notice also that if  $f_1$  is not  $f_{\text{end}}$ , then it is the first edge that takes the loop to a component of the domain that is no longer “visible” to  $f_{\text{end}}$ .

- (3) Suppose that  $f_n$  and  $L_1^T, L_2^T, \dots, L_n^T$  are known. If  $f_n$  is equal to  $f_{\text{end}}$ , stop and return the concatenation of  $L_1^T, L_2^T, \dots, L_n^T$  and  $f_n$  as the result  $T_w$ . Otherwise take the outward pointing  $e_n$  edge next to  $f_n$  and the loop  $L_{n+1}$  passing through  $e_n$ . Find the first edge of  $L_{n+1}$  after  $e_n$  in the orientation of  $L_{n+1}$  that lies in  $F_w$ . Call it  $f_{n+1}$  and the part of  $L_{n+1}$  between  $e_n$  and  $f_{n+1}$ , not including  $e_n$  and  $f_{n+1}$ ,  $L_{n+1}^T$ . Repeat (3).

The result of the algorithm  $T_w$  is called the *exploration process* from  $v_{\text{root}}$  to  $w$ . The collection  $\mathcal{T} = (T_w)_{w \in V_{\text{target}}}$  where  $V_{\text{target}} = V_\partial$  is called the *exploration tree* of the loop collection  $\mathcal{L}$  rooted at  $v_{\text{root}}$ .

The following result is immediate from the definition of the exploration tree. Basically, two coincide until they disconnect the target points and then they are independent.

**Proposition 2.2** (Target independence of exploration tree). *Suppose that  $v_{\text{root}}, w, w'$  are vertices in  $V_\partial$  in counterclockwise order, that can be the same. Let  $F_{w,w'}$  the edges*

of  $\partial_1\Omega$  that lie between  $w$  and  $w'$  in counterclockwise direction, including the edged at  $w$  and  $w'$ . Then  $T_w$  and  $T_{w'}$  are equal up until the first edge lying in  $F_{w,w'}$ .

Next we will construct the inverse of the loops-to-tree map which we will call a “tree-to-loops map.” The structure of the tree  $\mathcal{T}$  is the following: the branches of  $\mathcal{T}$  are simple and follow the rule of leaving white squares on their right and black on their left. The branching occurs in a subset of vertex set of  $\partial_1\Omega$ . There is a one-to-one correspondence between branching points of  $\mathcal{T}$  and the boundary touching loops of  $\mathcal{L}$ : the point on the loop, which lies on the boundary and is the closest one to the root if we move clockwise along the boundary, is a branching point and every branching point has this property for some loop. See also Figure 6. Suppose that at a branching point  $w$  the incoming edges are  $e_1$  and  $e_2$  and the outgoing edges are  $f_1$  and  $f_2$  and they are in the order  $e_1, f_2, e_2, f_1$  counterclockwise and that the exploration process enters  $w$  through  $e_1$ . Then necessarily each of the pairs  $e_1, f_1$  and  $e_2, f_2$  lie on the same loop of  $\mathcal{L}$  and these two loops are different. Also it follows that  $f_1$  and  $e_2$  are on  $\partial_1\Omega$  while  $f_2$  and  $e_1$  are not. It follows that the last edge of  $T_w$  is  $e_2$  and that the part of  $T_w$  between  $f_2$  and  $e_2$  is exactly the loop of  $\mathcal{L}$  that touches the boundary at  $f_w = e_2$ . Doing the same thing for every branching point defines a mapping from a suitable set of trees onto the set of loop collections of boundary touching loops of DCNIL. This mapping inverts the construction of the exploration tree and we summarize it in the following lemma.

**Lemma 2.3.** *The mapping from the collection of boundary touching loops  $\mathcal{L}_\partial$  to the chordal exploration tree  $\mathcal{T}$  is a bijection.*

Similar constructions work in the continuous setting. See [17] for the construction of the  $\text{SLE}(\kappa, \kappa - 6)$  exploration tree and the construction for recovering the loops.

Let us repeat here the second half of Theorem 1.1.

**Theorem 1.1 (b).** *The law of  $\Theta$  is given by the image of the  $\text{SLE}(\kappa, \kappa - 6)$  exploration tree with  $\kappa = 16/3$  under the above tree-to-loops mapping.*

### 3. TIGHTNESS OF TREES AND LOOP COLLECTIONS

In this section, we establish a priori bounds for trees and loop ensembles. The setting is relatively general, although we only apply it here to the FK Ising exploration tree of the boundary touching loops.

**3.1. A probability bound on multiple crossings by the tree.** Multiple crossings of annuli by random curves was considered in [1, 11]. Below we state the consequences for that type of result for the FK Ising exploration tree. Before that we make some crucial definitions.

We call a random variable  $X$  tight over a collection of probability measures  $\mathbb{P}$  on the probability space, if for each  $\varepsilon > 0$  there exists a constant  $M > 0$  such that  $\mathbb{P}(|X| < M) > 1 - \varepsilon$  for all  $\mathbb{P}$ . A crossing of an annulus  $A(z_0, r, R) = \{z \in \mathbb{C} : r < |z - z_0| < R\}$  is a closed segment of a curve that intersects both connected components of  $\mathbb{C} \setminus A(z_0, r, R)$  and a minimal crossing doesn't contain any genuine subcrossings.

Recall the general setup of [11]: we are given a collection  $(\phi, \mathbb{P}) \in \Sigma$  where the conformal map  $\phi$  contains also the information about its domain of definition  $(\Omega, v_{\text{root}}, w_0) = (\Omega(\phi), v_{\text{root}}(\phi), w_0(\phi))$  through the requirements

$$\phi^{-1}(\mathbb{D}) = \Omega, \quad \phi(v_{\text{root}}) = -1 \quad \text{and} \quad \phi(w_0) = 0$$

and  $\mathbb{P}$  is the probability law of FK Ising model on the discrete domain  $\Omega$  and in particular gives the distribution of the FK Ising exploration tree. Given the collection  $\Sigma$  of pairs  $(\phi, \mathbb{P})$  we define the collection  $\Sigma_{\mathbb{D}} = \{(\phi, \mathbb{P}) : (\phi, \mathbb{P}) \in \Sigma\}$  where  $\phi\mathbb{P}$  is the pushforward measure defined by  $(\phi\mathbb{P})(E) = \mathbb{P}(\phi^{-1}(E))$ .

**Theorem 3.1.** *The following claim holds for the collection of the probability laws of FK Ising exploration trees*

- for any  $\Delta > 0$ , there exists  $n \in \mathbb{N}$  and  $K > 0$  such that the following holds

$$\mathbb{P}(\text{at least } n \text{ disjoint segments of } \mathcal{T} \text{ cross } A(z_0, r, R)) \leq K \left(\frac{r}{R}\right)^{\Delta} \quad (6)$$

for all  $\mathbb{P} \in \Sigma_{\mathbb{D}}$ .

and there exist positive numbers  $\alpha, \alpha' > 0$  such that the following claims hold

- if for each  $r > 0$ ,  $M_r$  is the minimum of all  $m$  such that each  $T \in \mathcal{T}$  can be split into  $m$  segments of diameter less or equal to  $r$ , then there exists a random variable  $K(\mathcal{T})$  such that  $K$  is a tight random variable for the family  $\Sigma_{\mathbb{D}}$  and

$$M_r \leq K(\mathcal{T}) r^{-\frac{1}{\alpha}}$$

for all  $r > 0$ .

- All branches of  $\mathcal{T}$  can be jointly parametrized so that they are all  $\alpha'$ -Hölder continuous and the Hölder norm can be bounded by a random variable  $K'(\mathcal{T})$  such that  $K'$  is a tight random variable for the family  $\Sigma_{\mathbb{D}}$ .

Each of the claims have their own applications below although they are closely related, see [1].

*Proof.* We need to verify the first claim and the two other claims follow from it, by results of [1]. Notice that we need to verify the claim only for a subsequence of  $n$ 's, since the left-hand side of the inequality in the first claim is non-increasing in  $n$ .

Let  $A = A(z_0, r, R)$  be annulus,  $z_0 \in \mathbb{D}$ . Since either  $B(z_0, \sqrt{rR}) \subset \mathbb{D}$  or  $B(z_0, \sqrt{rR}) \cap \partial\mathbb{D} \neq \emptyset$ , it holds that we can choose  $r_1 = \sqrt{rR}$ ,  $R_1 = R$  or  $r_1 = r$ ,  $R_1 = \sqrt{rR}$  such that for  $A_1 = A(z_0, r_1, R_1)$ , either  $A_1 \subset \mathbb{D}$  or  $B(z_0, r_1) \cap \partial\mathbb{D} \neq \emptyset$ . For that  $A_1$  and for  $C > 1$  big enough, apply the estimate of conformal distortion given by either Lemma A.1 or Lemma A.2, depending on the case, to show that for any  $n = 1, 2, \dots, \lfloor \log(\frac{R}{r}) \rfloor$ , there exist  $\tilde{A}_m = A(z_m, r_m, 2r_m)$  such that the conformal image of any crossing of  $A(z_0, C^{n-1}r, C^n r)$  under  $\phi^{-1}$  is a crossing of  $\tilde{A}_m$ .

By Lemmas B.1 and B.2 in Appendix B and the results of [4] (in particular, Lemma 5.7) applied to crossings of  $\tilde{A}_m$ , it follows that for each  $\varepsilon > 0$  there is  $n$  such that  $\mathbb{P}(\text{at least } n \text{ disjoint segments of } \mathcal{T} \text{ cross } \tilde{A}_m) < \varepsilon$ . Thus (6) holds for  $n$  and constants  $K = \varepsilon^{-1}$  and  $\Delta = \log \frac{1}{\varepsilon}$ . Here the constant  $\Delta$  tends to  $\infty$  as  $n \rightarrow \infty$  (i.e. as  $\varepsilon$  tends to zero), and the estimates are uniform over all  $\mathbb{P} \in \Sigma_{\mathbb{D}}$  and annuli  $A(z_0, r, R)$  with  $R > r$ .  $\square$

**3.2. The crossing property of trees.** Let  $\gamma_k$ ,  $k = 0, \dots, N-1$ , is the collection  $\mathcal{T} = (T_x)$  in a (random) order and suppose that the random curves  $\gamma_k$  are each parametrized by  $[0, 1]$ . The chosen permutation specifies the order of exploration of the curves. More specifically, set  $\underline{\gamma}(t) = \gamma_k(t-k)$  when  $t \in [k, k+1)$ .

For a given domain  $\Omega$  and for a given simple (random) curve  $\gamma$  on  $\Omega$ , we set  $\Omega_\tau = \Omega \setminus \gamma[0, \tau]$  for each (random) time  $\tau$ . Similarly, for a given domain  $\Omega$  and for the given finite collection of curves  $\gamma_k$  on  $\Omega$ , we set  $\underline{\Omega}_\tau = \Omega \setminus \underline{\gamma}[0, \tau]$  for each (random) time  $\tau$ .

We call  $\Omega_\tau$  or  $\underline{\Omega}_\tau$  the domain at time  $\tau$ .

The following definition generalizes Definition 2.3 from [11].

**Definition 3.2.** For a fixed domain  $(\Omega, v_{\text{root}})$  and for fixed explored collection  $\underline{\gamma}$  of curves  $T_x$ ,  $x \in V_{\text{target}}$ , where each curve  $\gamma_x$  is contained in  $\overline{\Omega}$ , starting from  $v_{\text{root}}$  and ending at a point  $x$  in the set  $V_{\text{target}}$ , define for any annulus  $A = A(z_0, r, R)$ , for every (random) time  $\tau \in [0, N]$  and  $x \in V_{\text{target}}$ ,  $A_\tau^{u,x} = \emptyset$  if  $\partial B(z_0, r) \cap \partial \underline{\Omega}_\tau = \emptyset$  and

$$A_\tau^{u,x} = \left\{ z \in \underline{\Omega}_\tau \cap A : \begin{array}{l} \text{the connected component of } z \text{ in } \underline{\Omega}_\tau \cap A \\ \text{doesn't disconnect } \gamma(\tau) \text{ from } x \text{ in } \underline{\Omega}_\tau \end{array} \right\} \quad (7)$$

otherwise. Define also

$$A_\tau^{f,x} = \left\{ z \in \Omega_\tau \cap A : \begin{array}{l} \text{the connected component of } z \text{ in } \underline{\Omega}_\tau \cap A \\ \text{is crossed by any path connecting } \gamma(\tau) \text{ to } x \text{ in } \underline{\Omega}_\tau \end{array} \right\} \quad (8)$$

and set  $A_\tau^u = \bigcap_{x \in V_{\text{target}}} A_\tau^{u,x}$  and  $A_\tau^f = \bigcup_{x \in V_{\text{target}}} A_\tau^{f,x}$ . We say that  $A_\tau^{u,x}$  is *avoidable* for  $\gamma_x$  and  $A_\tau^u$  is *avoidable for all* (branches). We say that  $A_\tau^{f,x}$  is *unavoidable* for  $\gamma_x$  and  $A_\tau^f$  is *unavoidable for at least one* (branch). Here and in what follows we only consider allowed lattice paths when we talk about connectedness.

*Remark 3.3.* Note that  $A_\tau^u \cap A_\tau^f = \emptyset$ . This follows from  $A_\tau^{u,x} \cap A_\tau^{f,x} = \emptyset$  which holds by definition.

Recall that  $\partial\Omega$  is the boundary of  $\Omega$  and  $\partial_1\Omega$  is the internal boundary of  $\Omega$ , that is, the (allowed) lattice path inside  $\Omega$  which stays as close as possible to  $\partial\Omega$ . A point on  $\partial_1\Omega$  is a *branching point* of the tree  $\mathcal{T}$  if it is the last common point of two branches. In that case the edge on the primal lattice passing through the point has to be open in the random cluster configuration. See also Figure 7.

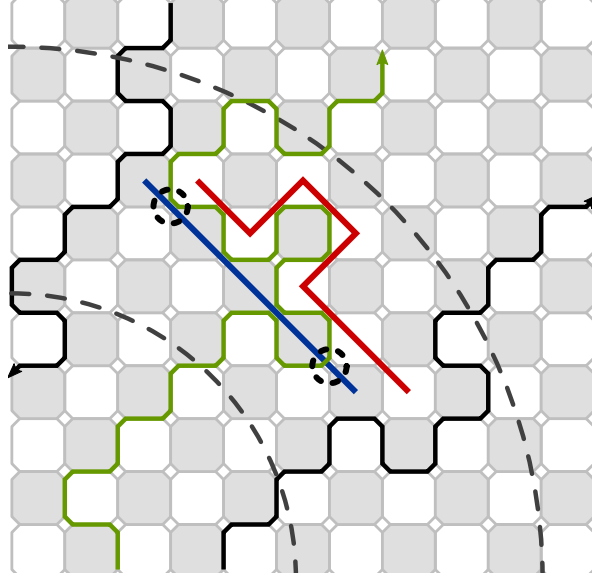


FIGURE 7. The region between the large dashed circular arcs is a quarter of an annulus. A crossing of an annulus is drawn in green and the boundaries of the domain  $\underline{\Omega}_\tau$  in black and they are oriented according to the orientation of the medial lattice. As indicated by the figure if there are transversal open and dual-open paths in the annulus, then the crossing has to go near the left- and right-hand boundaries of the annular sector.

Next we will write down a required estimate in the form of a hypothesis analogous the ones presented in Section 2 of [11]. In fact, we will present two equivalent

conditions here. As we will later see that conformal invariance will hold for this type of conditions, again analogously to [11]. These conditions will be verified for the FK Ising exploration tree below.

**Condition G1.** Let  $\Sigma$  be as above. If there exists  $C > 1$  such that for any  $(\phi, \mathbb{P}) \in \Sigma$ , for any stopping time  $0 \leq \tau \leq N$  and for any annulus  $A = A(z_0, r, R)$  where  $0 < Cr \leq R$ , it holds that

$$\mathbb{P} \left( \begin{array}{c} \underline{\gamma}[\tau, N] \text{ makes a crossing of } A \text{ which is contained in } \overline{A_\tau^u} \\ \text{or} \\ \text{which is contained in } \overline{A_\tau^f} \text{ and the first minimal crossing} \\ \text{doesn't have branching points on both sides} \end{array} \middle| \mathcal{F}_\tau \right) < \frac{1}{2}. \quad (9)$$

then the family  $\Sigma$  is said to satisfy a *geometric joint unforced–forced crossing bound*. Call the event above  $E^{\text{u,f}}$ .

See Figure 8 for more information about different types of branching points.

**Condition G2.** The family  $\Sigma$  is said to satisfy a *geometric joint unforced–forced crossing power-law bound* if there exist  $K > 0$  and  $\Delta > 0$  such that for any  $(\phi, \mathbb{P}) \in \Sigma$ , for any stopping time  $0 \leq \tau \leq N$  and for any annulus  $A = A(z_0, r, R)$  where  $0 < r \leq R$ ,

$$\text{LHS} \leq K \left( \frac{r}{R} \right)^\Delta. \quad (10)$$

Here LHS is the left-hand side of (9).

We want to use Condition G1 or equivalently G2 as a hypothesis for theorems. We start by verifying them for the critical FK Ising model exploration tree.

**Theorem 3.4.** *If  $\Sigma$  is the collection of pairs  $(\phi, \mathbb{P})$  where  $\phi$  satisfies the above properties and also that  $U(\phi)$  is a discrete domain with some lattice mesh, and  $\mathbb{P}$  is the law of the critical FK Ising model exploration tree  $\mathcal{T}$  on  $U(\phi)$ , then  $\Sigma$  satisfies Conditions G1 and G2.*

*Proof.* The theorem can be proved in the same way as the result that a single interface in FK Ising model satisfies a similar condition, which was presented in Section 4.1 of [11]. However, we need stronger crossing estimates, which were established in Theorem 1.1 of [4]. Basically those crossing estimates hold for a general topological quadrilateral with general boundary conditions. The types of boundary conditions that we need to consider are shown in Figure 9(b). By the FKG inequality, we get a lower bound on such crossing event if we change the boundary conditions along the dotted arcs (the boundary of the annulus) to the color opposite to the color of the crossing, in Figure 9(b). Those are ultimately the topological quadrilaterals to which we apply Theorem 1.1 of [4].  $\square$

As shown in [11], this type of bounds behave well under conformal maps. We have uniform control on how the constants in Conditions G1 and G2 change if we transform the random objects conformally from one domain to another.

Given a collection  $\Sigma$  of pairs  $(\phi, \mathbb{P})$  we define the collection  $\Sigma_{\mathbb{D}} = \{(\phi\mathbb{P}) : (\phi, \mathbb{P}) \in \Sigma\}$  where  $\phi\mathbb{P}$  is the pushforward measure defined by  $(\phi\mathbb{P})(E) = \mathbb{P}(\phi^{-1}(E))$ .

**Theorem 3.5.** *If  $\Sigma$  is as in Theorem 3.4, then  $\Sigma_{\mathbb{D}}$  satisfies Conditions G1 and G2.*

### 3.3. Uniform approximation by finite subtrees.

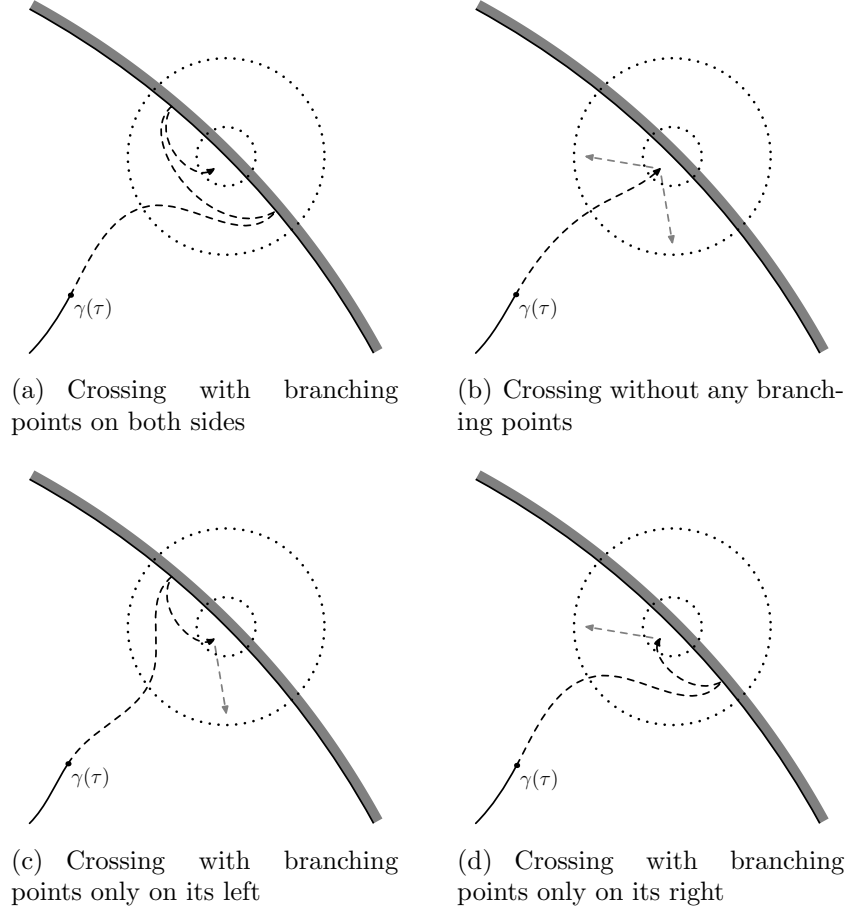


FIGURE 8. Condition G1 or G2 imply that the crossing events of any of the types illustrated in the figures (b), (c) and (d) has small probability. The longer black arrow is the crossing event considered in (9) and the shorter gray arrows are the crossings of the annulus that are still possible afterwards.

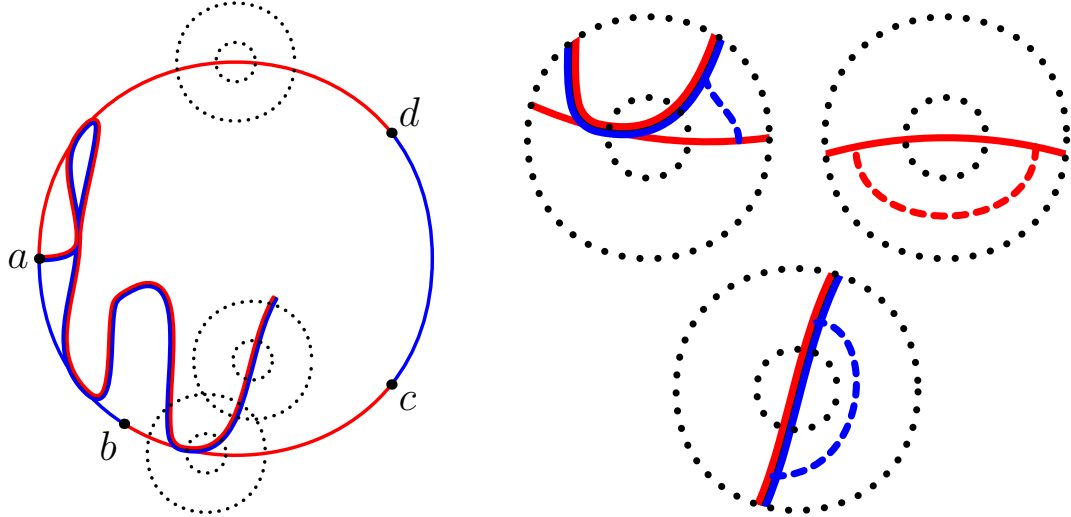
3.3.1. *Topology on trees.* Let  $d_{\text{curve}}$  to be the curve distance (defined to be the infimum over all reparametrizations of the supremum norm of the difference). We define the space of trees as the space of closed subsets of the space of curves and we endow this space with the Hausdorff metric. More explicitly, if  $(T_x)_{x \in S}$  and  $(\hat{T}_{\hat{x}})_{\hat{x} \in \hat{S}}$  are trees then the distance between them is given by

$$d_{\text{tree}} \left( (T_x)_{x \in S}, (\hat{T}_{\hat{x}})_{\hat{x} \in \hat{S}} \right) = \max \left\{ \sup_x \inf_{\hat{x}} d_{\text{curve}} (T_x, \hat{T}_{\hat{x}}), \sup_{\hat{x}} \inf_x d_{\text{curve}} (T_x, \hat{T}_{\hat{x}}) \right\}. \quad (11)$$

3.3.2. *Uniform approximation by finite subtrees.* Let us divide the boundary of the unit disc into a finite number of connected arcs  $I_k$ ,  $k = 1, \dots, N$ , which are disjoint, except possibly at their end points. Let  $x_k^\pm$  be the end points of  $I_k$ . Suppose that  $-1 = \phi(v_{\text{root}})$  is an end point. Naturally it is then an end point of two arcs.

Let  $\mathcal{I} = \{I_k : k = 1, \dots, N\}$  and denote the maximum of the diameters of  $I_k$  by  $m(\mathcal{I})$  and  $\{x_k^\pm : k = 1, \dots, N\}$  by  $\hat{\mathcal{S}}^\mathbb{D}$ . Consider now the random tree  $\mathcal{T}_\delta = (T_{v_{\text{root}}, x})_x$  whose law is given by  $\mathbb{P}_\delta$ . The finite subtree  $\mathcal{T}_\delta(\mathcal{I})$  is defined by the following steps:

- first take the discrete approximation  $\hat{\mathcal{S}}_\delta$  of the set of points  $\phi^{-1}(\hat{\mathcal{S}}^\mathbb{D})$  on the vertex set of  $\partial_1 \Omega_\delta$ .



(a) The different types of locations of annuli needed to be considered in Theorem 3.4.

(b) For each annulus type, we need to establish a probability lower bound on the crossing events of connected random cluster paths indicated by the dashed lines to get the bound of Condition G2.

FIGURE 9. Crossing estimates needed for Theorem 3.4. The setup with four marked points is the same as in Figure 10 in Section 4.

- then consider the finite subtree  $\hat{\mathcal{T}}_\delta = (T_{v_{\text{root}},x})_{x \in \hat{\mathcal{S}}_\delta}$
- finally set  $\mathcal{S}_\delta$  to be the union of  $\hat{\mathcal{S}}_\delta$  and all the branching points (in the sense of the definition in Section 3.2) of  $\hat{\mathcal{T}}_\delta$ . Denote  $(T_{v_{\text{root}},x})_{x \in \mathcal{S}_\delta}$  by  $\mathcal{T}_\delta$ . Here  $T_{v_{\text{root}},x}$  for a branching point is defined as the subpath of  $(T_{v_{\text{root}},x})_{x \in \hat{\mathcal{S}}_\delta}$  that starts from  $v_{\text{root}}$  and terminates at  $x$ .

In other words, we take the subtree corresponding to  $\hat{\mathcal{S}}$  and then we augment it by adding all its branching points and the branches ending at those branching points to the tree. We will call below  $\mathcal{T}_\delta = \mathcal{T}_\delta(\mathcal{I})$  the *finite subtree* corresponding to  $\mathcal{I}$ . It is finite in a uniform way over the family of probability laws and domains of definition. Hence the name.

Denote the image of  $\mathcal{T}_\delta(\mathcal{I})$  under  $\phi$  by  $\mathcal{T}_\delta^{\mathbb{D}}(\mathcal{I})$ .

**Theorem 3.6.** *For each  $R > 0$*

$$\sup_{\mathbb{P}_\delta} \mathbb{P}_\delta \left[ d_{\text{tree}}(\mathcal{T}_\delta^{\mathbb{D}}(\mathcal{I}), \mathcal{T}_\delta^{\mathbb{D}}) > R \right] = o(1) \quad (12)$$

as  $m(\mathcal{I}) \rightarrow 0$ .

*Proof.* Let  $\varepsilon > 0$  and suppose that  $K$  and  $\alpha$  are positive numbers such that  $M_r \leq Kr^{-\frac{1}{\alpha}}$  for all  $r > 0$  with probability greater than  $1 - \varepsilon$  for any  $\mathbb{P}_\delta$ . Such constants exist by Theorem 3.1.

Let  $x \in V_{\text{target}}$ . Take  $k$  such that  $x \in I_k$ . Set  $x_\pm = x_k^\pm$ . We will show that with high probability  $T_{v_{\text{root}},x}$  is close to either  $T_{v_{\text{root}},x_-}$ ,  $T_{v_{\text{root}},x_+}$  or some  $T_{v_{\text{root}},\tilde{x}}$  where  $\tilde{x}$  is a branch point on the arc  $x_-x_+$ . We will show that with a high probability this happens for all  $x$  simultaneously.

Fix  $R > 0$ . We will study under which circumstances

$$\min \{d(T_{v_{\text{root}},x}, T_{v_{\text{root}},\tilde{x}}) : \tilde{x} \in x_-x_+\} \quad (13)$$

is larger than  $R$  or smaller or equal to  $R$ .

First we notice, that if  $x_0$  the first branching point on  $I_k$ , then the branches to  $x, x_-$  and  $x_+$  are all the same until they reach  $x_0$ .

We will consider two complementary cases: either  $x_0 \in x_-x$  or  $x_0 \in xx_+$ .

If  $x_0 \in x_-x$ , the branches of  $x$  and  $x_+$  continue to be the same after  $x_0$  at least for some time. Suppose that  $\min\{d(T_{v_{\text{root}},x}, T_{v_{\text{root}},x_0}), d(T_{v_{\text{root}},x}, T_{v_{\text{root}},x_+})\} > R$ . Then in particular the branch that we follow to continue towards  $x_-$  and  $x_0$  has to reach distance  $3R/4$  from  $x_-x_+$  without making a branch point in  $xv_{\text{root}}$ . It has to contain therefore a subpath which has diameter at least  $R/2$  and whose other end is at the boundary and which is otherwise disjoint from the boundary. Call the subpaths with this property  $\mathcal{J}$ . Then by the bound on  $M_r$  we get  $\#\mathcal{J} \leq K(R/8)^{-\frac{1}{\alpha}}$ .

If we happen to reach distance  $3R/4$  from  $x_-x_+$  without making a branch point, then in order  $d(T_{v_{\text{root}},x}, T_{v_{\text{root}},x_+}) > R$  to hold the path needs to come close to  $x_-x_+$  again and branch on  $x_0x_+$  which it will do surely. But while doing so one of the following has to hold:

- the returning path can't branch on  $x_-v_{\text{root}}$  before branching in  $x_0x_+$  and then the continuation to  $x_+$  has to reach distance  $3R/4$  from  $x_-x_+$
- or the returning path branches in  $x_0x_+$  and then the continuation to  $x$  reaches distance  $3R/4$  from  $x_-x_+$ .

By the conditions presented in the previous section the probabilities of both of them can be made less than  $\varepsilon$  by choosing length of  $x_-x_+$  small.

Now there are  $\#\mathcal{J}$  subpaths where this can happen hence the total probability is at most

$$\varepsilon + K(R/8)^{-\frac{1}{\alpha}}\varepsilon = \text{cst.}(R)\varepsilon$$

which can be made arbitrarily small. The other case  $x_0 \in xx_+$  is similar.  $\square$

### 3.4. Precompactness of the loops and recovering them from the tree.

3.4.1. *Topology for loops.* Let  $\mathbb{T}$  be the unit circle. We consider a loop to be a continuous function on  $\mathbb{T}$ , considered modulo orientation preserving reparametrizations of  $\mathbb{T}$ . The metric on loops is given in a similar fashion as for curves. We define

$$d_{\text{loop}}(\gamma, \hat{\gamma}) = \inf \left\{ \|f - \hat{f}\|_{\infty} : f \in \gamma, \hat{f} \in \hat{\gamma} \right\}. \quad (14)$$

The space of loops endowed with  $d_{\text{loop}}$  is a complete and separable metric space.

A loop collection  $\Theta$  is a closed subset of the space of loops. On the space of loop collections we use the Hausdorff distance. Denote that metric by  $d_{\text{LE}}$ , where LE stands for *loop ensemble*.

The random loop configuration we are considering is the collection of FK Ising loops which we orient in clockwise direction.

#### 3.4.2. Precompactness of the loops.

**Theorem 3.7.** *The family of probability laws of  $\Theta$  is tight in the metric space of loop collections.*

*Proof.* Directly follows from Lemma 2.3 and the tightness of the trees. Namely, observe that if the random tree can be split into a tight random number of path segments of diameter less than  $R$ , then that cover can be used for covering of the loops.  $\square$



**3.5. Uniform approximation of loops by the finite subtrees.** By Lemma 2.3 it is possible to reconstruct the loops from the full tree. And by Theorem 3.6 we can approximate the full tree by finite subtrees. How do we recover loops approximately from the finite subtrees? Can we do it in a uniform manner?

Consider a loop  $\theta \in \Theta$ . We divide it into *arcs* which are excursions from the boundary to the boundary, otherwise disjoint from the boundary. Since the loop is oriented in the clockwise direction, exactly one of the arcs goes away from  $v_{\text{root}}$  in the counterclockwise orientation of the boundary and rest of them go towards  $v_{\text{root}}$ . The first case is the *top arc*  $\gamma^T$  of the loop  $\gamma$  and the rest of them are the *bottom arcs*. See Figure 6. The concatenation of the bottom arcs is denoted by  $\gamma^B$ .

Suppose that the loop doesn't intersect a fixed neighborhood of  $v_{\text{root}}^{\mathbb{D}}$ . The diameter of the entire loop is bounded by a uniform constant (depending on the chosen neighborhood) times the diameter of the top arc and vice versa. This means that if the loop has diameter greater than a fixed number, then the top arc is traced by finite subtree for fine enough mesh according to Theorem 3.6.

Also the bottom arcs from right to left are traced until the last branch point to the points defining the finite subtree. Again by Theorem 3.6, the part which remains to be discovered of the loops, has small diameter. Hence if we define approximate loop to have just a linear segment in that place, we see that the loop and the approximation are close in the given metric.

So we define the finite-subtree approximation  $\Theta_\delta$  of the random loop collection  $\Theta$  to be the collection of all those discovered top arcs concatenated with their discovered bottom arcs and the linear segment needed to close the loop. By Theorem 3.6, we have the following result.

**Theorem 3.8.** *For each  $R > 0$*

$$\sup_{\delta} \mathbb{P}_{\delta} [\text{d}_{\text{LE}}(\Theta_{\delta}^{\mathbb{D}}(\mathcal{I}), \Theta^{\mathbb{D}}) > R] = o(1) \quad (15)$$

as  $m(\mathcal{I}) \rightarrow 0$ .

**3.6. Some a priori properties of the loop ensembles.** The following theorem gathers some technical estimates needed below.

**Theorem 3.9.** *The family of critical FK Ising loop ensemble measures  $(\mathbb{P}_{\delta})_{\delta>0}$  satisfy the following properties*

(a) *(finite number of big loops) For each  $R > 0$ ,*

$$\sup_{\delta>0} \mathbb{P}_{\delta} (\# \{ \gamma : \text{diam}(\gamma) \geq R \} \geq N) = o(1) \quad (16)$$

as  $N \rightarrow \infty$ .

(b) *(small loops and branching points are dense on the boundary) There exists  $\Delta > 0$  such that for each  $x \in \partial\mathbb{D}$  and  $R > 0$*

$$\sup_{\delta>0} \mathbb{P}_{\delta} (\sup \{ \text{diam}(\gamma) : \gamma \cap B(x, r) \neq \emptyset \} \geq R) = \mathcal{O} \left( \left( \frac{r}{R} \right)^{\Delta} \right) \quad (17)$$

as  $r \rightarrow 0$ .

(c) *(big loops have positive support on the boundary and they touch the boundary infinitely often around the extremal points) For any  $R > 0$*

$$\sup_{\delta>0} \mathbb{P}_{\delta} (\exists \gamma \text{ s.t. } \text{diam}(\gamma) \geq R \text{ and } |x_L(\gamma) - x_R(\gamma)| < r) = o(1) \quad (18)$$

as  $r \rightarrow 0$ . Here  $x_L(\gamma)$  and  $x_R(\gamma)$  are the left-most and the right-most points of  $\gamma$  along the boundary of the domain. We call the boundary arc  $x_L(\gamma)x_R(\gamma)$

the support of  $\gamma$  on the boundary. Furthermore, for any constant  $0 < \eta < 1$  there exists a sequence  $\delta_0(m) > 0$  and constant  $\lambda > 0$  such that if  $x(\gamma) = \gamma^L$  or  $x(\gamma) = \gamma^R$

$$\sup_{\delta \in (0, \delta_0(m))} \mathbb{P}_\delta \left( \begin{array}{c} \exists \gamma \text{ s.t. } \text{diam}(\gamma) \geq R \text{ and } |\gamma^L - \gamma^R| \geq r : \\ \#\{n = 1, 2, \dots, m : \gamma \text{ touches boundary} \\ \text{in } A(x(\gamma), \eta^n r, \eta^{n-1} r)\} \leq \lambda m \end{array} \right) = o(1) \quad (19)$$

as  $m \rightarrow \infty$ .

(d) (big loops are not pinched)

$$\sup_{\delta > 0} \mathbb{P}_\delta \left( \begin{array}{c} \exists \gamma \text{ s.t. } \gamma^T = \gamma_1^T \sqcup \gamma_2^T, \gamma_1^T \cap \gamma_2^T = \{x\} \\ \text{diam}(\gamma_k^T) \geq R \text{ for } k = 1, 2 \text{ and } \text{dist}(x, \partial \mathbb{D}) < r \end{array} \right) = o(1) \quad (20)$$

and

$$\sup_{\delta > 0} \mathbb{P}_\delta \left( \begin{array}{c} \exists \gamma \text{ and } \gamma' \subset \gamma^B \text{ s.t. } \gamma' = \gamma'_1 \sqcup \gamma'_2, \gamma'_1 \cap \gamma'_2 = \{x\} \\ \gamma' \cap \partial_1 \Omega = \emptyset, \text{diam}(\gamma'_k) \geq R \text{ for } k = 1, 2 \text{ and } \text{dist}(x, \partial \mathbb{D}) < r \end{array} \right) = o(1) \quad (21)$$

as  $r \rightarrow 0$ .

(e) (big loops are distinguishable)

$$\sup_{\delta > 0} \mathbb{P}_\delta (\exists \gamma, \tilde{\gamma} \text{ s.t. } \text{diam}(\gamma), \text{diam}(\tilde{\gamma}) \geq R \text{ and } d_{\text{loop}}(\gamma, \tilde{\gamma}) < r) = o(1) \quad (22)$$

as  $r \rightarrow 0$ .

*Proof.* The bound (16) follows from Theorem 3.6. The bound (17) follows directly from the crossing bound of annuli at  $x$ .

The bound (19) is shown to hold by considering the domain  $U \setminus \underline{I}(\tau)$  where  $\tau$  is a stopping time such that the top arc of a big loop  $\gamma$  is reaching its endpoint  $x(\gamma)$  at time  $\tau$ , and using the crossing bound of annuli at  $x(\gamma)$ .

The proofs of (18), (20) and (21) are similar. Use the exploration process which discovers the top arc of any loop before the lower arcs. As noted before, the diameter of the top arc is comparable to the diameter of the whole loop. Hence it enough to work with loops that have top-arc diameter more than  $R$ .

For (18), stop the process when the current arc exits the ball of radius  $R/4$  around the starting point  $z_0$  of the arc. The rest of the exploration process makes with high probability a branching point in the annulus  $A(z_0, r, R/4)$  before finishing the loop at  $z_0$ . This implies that the event (18) does not occur for that loop. These segments of diameter  $R/4$  in the exploration process are necessarily disjoint (they start on the boundary and remain after that disjoint from the boundary) and hence their number is a tight random variable by Theorem 3.1. The bound (18) follows by a simple union bound.

For (20), stop the process when the diameter of the current arc is at least  $R$  and the tip lies within distance  $r$  from the boundary. Let the point closest on the boundary be  $z_0$ . These arc of diameter  $R$  are necessarily disjoint: they start from the boundary and remain after that disjoint from the boundary. The rest of the exploration makes with high probability a branching point before it exits the ball of radius  $R$  centered at  $z_0$ . The bound (20) follows by a simple union bound.

The proof of (21) is very similar and we omit the details.

For the bound (22), let  $\gamma_1$  and  $\gamma_2$  be two loops of diameter larger than  $R$ . Let  $S$  be the set of points of  $\gamma_2$  and let  $\gamma_1^T$  be the top arc of  $\gamma_1$  and let  $\gamma_1^B$  be the rest of  $\gamma_1$ . Without loss of generality, we can assume that  $\gamma_1^T$  separates  $S$  from  $\gamma_1^B$  in  $\mathbb{D}$ .

Otherwise we can exchange the roles of  $\gamma_1$  and  $\gamma_2$ . Suppose that  $d_{\text{loop}}(\gamma_1, \gamma_2) < r$ . Then we claim that the Hausdorff distance of  $\gamma_1^T$  and  $\gamma_1^B$  is less than  $2r$ . The distance from any point of  $\gamma_1^B$  to  $\gamma_1^T$  is less than  $r$ . This follows when we notice that any line segment connecting  $\gamma_1^-$  to  $S$  intersects  $\gamma_1^+$ . Due to topological reasons, since  $\gamma_1$  and  $\gamma_2$  are oriented and  $d_{\text{loop}}$  uses this orientation, it also holds that the distance from any point of  $\gamma_1^T$  to  $\gamma_1^B$  is less than  $2r$ . Namely, any point in  $\gamma_1^T$  has a point of  $\gamma_2$  in its  $r$  neighborhood, which has distance less than  $r$  to  $\gamma_1^B$ . We can now use the exploration process to explore  $\gamma_1^T$ . Take a point  $z_0 \in \gamma_1^T$  that has distance at least  $\sqrt{rR}$  to the boundary. Such a point exists by (18) and (21). Now by the crossing property, with high probability the exploration of  $\gamma_1^B$  doesn't come close to  $z_0$ .  $\square$

**3.6.1. Some consequences.** In the discrete setting we are given a tree-loop ensemble pair. The tree and the loop ensemble are in one to one correspondence as explained earlier. Recall that, given the loop ensemble, the tree is recovered by the exploration process which follows the loops in counterclockwise direction and jumps to the next loop at boundary points where the loop being followed turns away from the target point. Recall also that the loops are recovered from the tree by noticing that the leftmost point in the loop corresponds to a branching point of the tree and the rest of the loop is the continuation of the branch to the point just right of that branching point.

Take a subsequence of the sequence of the tree-loop ensemble pairs that converges weakly. We can choose a probability space so that they converge almost surely. The collection of loops is almost surely at most countable also in the limit. By the properties of the loop ensemble given in Theorem 3.9, the limiting pair and the process of taking the limit have the following properties

- the loops are distinguishable in the sense that there is no sequence of pairs of distinguishable loops that would converge to the same loop.
- Each loop consists of a single top arc which is disjoint from the boundary except at the endpoints and a non-empty collection of bottom arcs. In particular, the endpoints of the top arc (the leftmost and rightmost points of the loop) are different.

From these properties we can prove the following result. The second assertion basically means that there is a way to *reconstruct the loops from the tree* also in the limit.

**Theorem 3.10.** *Let  $(\Theta^{\mathbb{D}}, \mathcal{T}^{\mathbb{D}})$  be the almost sure limit of  $(\Theta_{\delta_n}^{\mathbb{D}}, \mathcal{T}_{\delta_n}^{\mathbb{D}})$  as  $n \rightarrow \infty$ . Write  $\Theta^{\mathbb{D}} = (\theta_j)_{j \in J}$  and  $\Theta_{\delta_n}^{\mathbb{D}} = (\theta_{n,j})_{j \in J}$  (with possible repetitions) such that almost surely for all  $j \in J$ ,  $\theta_{n,j}$  converges to  $\theta_j$  as  $n \rightarrow \infty$ , and then set  $x_{n,j}$  to be the target point of the branch of  $\mathcal{T}_{\delta_n}^{\mathbb{D}}$  that corresponds to  $\theta_{n,j}$  in the above bijection (described in the beginning of the subsection 3.6.1). Then*

- *Almost surely all  $x_{n,j}$  converge to some points  $x_j$  as  $n \rightarrow \infty$  and all the branches  $T_{x_{n,j}}^+$  converge to some branches denoted by  $T_{x_j}^+$  as  $n \rightarrow \infty$ . Furthermore,  $x_j$  are distinct and they form a dense subset of  $\partial\mathbb{D}$  and  $\mathcal{T}$  is the closure of  $(T_{x_j}^+)_{j \in J}$ .*
- *On the other hand,  $(T_{x_j}^+)_{j \in J}$  is characterized as being the subset of  $\mathcal{T}$  that contains all the branches of  $\mathcal{T}$  that have a doublepoint on the boundary. Furthermore, that doublepoint is unique and it is the target point (that is, endpoint) of that branch. Any loop  $\theta_j$  can be reconstructed from  $(T_{x_j}^+)_{j \in J}$  in the following way: the leftmost point of  $\theta_j \cap \partial\mathbb{D}$  (the point closest to the root*

in the counterclockwise direction) is one of the doublepoints  $x_j^+$  and the loop  $\theta_j$  is the part between the first and last visit to  $x_j^+$  by  $T_{x_j^+}$ .

### 3.7. Precompactness of branches and description as Loewner evolutions.

3.7.1. *Precompactness of a single branch.* We know now that the sequence of exploration trees is tight in the space of curve collections, which has the topology of Hausdorff distance on the compact sets of the space of curves. This enables us to choose for any subsequence a convergent subsequence. However, it turns out that we need stronger tools to be able to characterize the limit. We will review the results of [11] that we will use.

The hypothesis of [11] is similar to the conditions in Section 3.2. Once that hypothesis holds for a sequence of random curves, it is shown in that paper that the sequence is tight in the topology of the space of curves. Furthermore, it is established that such a sequence is also tight in the topology of uniform convergence of driving terms of Loewner evolutions in such a way that mapping between curves and Loewner evolutions is uniform enough so that if a sequence converges in both of the above mentioned topologies the limits have to be the same.

The hypothesis of [11] for the FK Ising branch has been already established since we can see it as a special case of Theorem 3.4.

**Theorem 3.11** (Kempainen–Smirnov, [11]). *Under certain hypothesis, a sequence probability laws  $\mathbb{P}_n$  of random curves of  $\mathbb{H}$  has the following properties: for each  $\varepsilon > 0$ , there exists an event  $K$  such that  $\inf_n \mathbb{P}_n(K) \geq 1 - \varepsilon$  and the capacity-parametrized curves in  $K \cap \{\gamma \text{ simple}\}$  form an equicontinuous family, their driving processes form an equicontinuous family and finally  $|\gamma(t)| \rightarrow \infty$  as  $t \rightarrow \infty$  uniformly. Moreover the driving processes on the event  $K \cap \{\gamma \text{ simple}\}$  are  $\beta$ -Hölder continuous with a bounded Hölder constant for any  $\beta \in (0, \frac{1}{2})$ .*

In addition to [11], see also Section 6.3 in [10] for this type of argument in the case of site percolation.

3.7.2. *Precompactness of finite subtrees.* For fixed finite number of curves it is straightforward to generalize Theorem 3.11. In fact, the conclusions of Theorem 3.11 hold for any finite subtree that we considered in Section 3.5.

In the rest of the paper we use these tools available for us and aim to characterize the scaling limits of finite subtrees of the exploration tree. If we manage to establish the uniqueness of the subsequent scaling limit of those objects, then Theorem 1.1 follows from tightness and from the approximation result, Theorem 3.8.

## 4. PREHOLOMORPHIC MARTINGALE OBSERVABLE

4.1. **The setup for the observable.** It is natural to generalize the setup of the previous section to domains of type illustrated in Figure 10.

Henceforth we consider  $G^\spadesuit$  with four special boundary vertices  $a, b, c, d$ , where the boundary edges (when consider as edges of the directed graph  $G_\rightarrow^\spadesuit \subset \mathbb{L}_\rightarrow^\spadesuit$ ) at  $a$  and  $c$  point inwards and the boundary edges at  $b$  and  $d$  outwards. Now  $a$  and  $d$  play the roles of  $v_{\text{root}}$  and  $w$ , respectively, in the construction of the exploration tree of the previous section. The arcs  $ab$  and  $cd$  have white boundary ( $\mathbb{L}^\circ$  wired) and  $bc$  and  $da$  have black boundary ( $\mathbb{L}^\bullet$  wired).

To get a random cluster measure on  $G^\spadesuit$  consistent with the all wired boundary conditions and the exploration process in Section 2, we have to count the wired arcs  $bc$  and  $da$  to be in the same cluster. This corresponds to the external arc

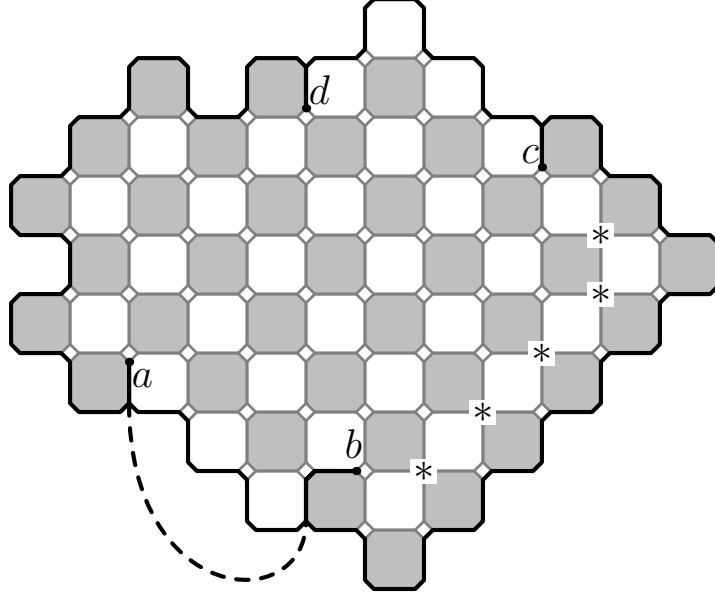


FIGURE 10. Generalized setting for the chordal exploration tree. We add an external arc from  $b$  to  $a$  with the following interpretation. The observable contains an indicator factor for the curve starting from  $c$  and a complex factor depending on the winding of that curve. If that curve goes to  $b$  we continue to follow it through the external arc to  $a$  and from there to  $d$ . On the other hand, if the curve starting from  $c$  ends directly to  $d$ , then the curve connecting  $a$  to  $b$  is counted as a loop giving an additional weight  $\sqrt{2}$  to the configuration. The arc  $bc \subset V_{\partial,1}$  is marked with  $*$ 's.

configuration where  $a$  and  $b$  are connected by an arc and  $c$  and  $d$  are connected by an arc. Denote such *external arc pattern*  $(a \smile b, c \smile d)$ . Later we will denote *internal arc patterns* by  $(a \frown b, c \frown d)$  etc.

In fact, we will choose not to draw the external arc  $c \smile d$ . The reason for this is that it is not used in the definition of the observable and the weights for loop configurations that we get either with or without  $c \smile d$  are all proportional by the same  $\sqrt{2}$  factor. Thus it doesn't change the probability distribution.

The configuration on  $G^\spadesuit$  is  $(\gamma_1, \gamma_2, l_i : i = 1, 2, \dots, N_{\text{loops}})$ , where  $\gamma_1$  and  $\gamma_2$  are the paths starting from  $a$  and  $c$ , respectively. Define

$$\hat{\gamma} = \begin{cases} \gamma_2 & \text{if } \gamma_2 \text{ exits through } d \\ \gamma_2 \sqcup \alpha \sqcup \gamma_1 & \text{if } \gamma_2 \text{ exits through } b \end{cases} \quad (23)$$

where  $\alpha$  is the planar curve that realizes the exterior arch  $a \smile b$  as in Figure 10 and  $\sqcup$  denotes the concatenation of paths. The first case in (23) is the internal arc pattern  $(a \frown b, c \frown d)$  and the second is  $(a \frown d, c \frown b)$ .

For a sequence of domains  $G_\delta^\spadesuit \subset \delta\mathbb{L}^\spadesuit$  with  $a_\delta, b_\delta, c_\delta, d_\delta$ , define the observable as

$$f_\delta(e) = \theta_\delta \mathbb{E}_\delta(\mathbb{1}_{e \in \hat{\gamma}} e^{-i\frac{1}{2}W(d_\delta, e)}) \quad (24)$$

for any  $e \in E(G_\delta^\circ) \subset E(G_\delta^\spadesuit)$ . Here  $W(d_\delta, e)$  is the winding from boundary edge of  $d_\delta$  to  $e$  along the reversal of  $\hat{\gamma}$  and  $\theta_\delta$  (satisfying  $|\theta_\delta| = 1$ ) is a constant, whose value we specify later. Notice that  $f_\delta$  doesn't depend on the choice of  $\alpha$  since the winding is well-defined modulo  $4\pi$ .

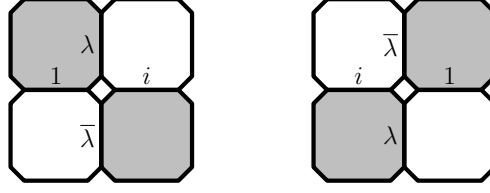


FIGURE 11. The associated lines  $\mathbb{R}, i\mathbb{R}, \lambda\mathbb{R}, \bar{\lambda}\mathbb{R}$  around the two different types of vertices of  $\mathbb{L}^\diamond$ . Note that for a directed edge  $e \in \mathbb{L}_{\rightarrow}^\diamond$ , the line in the complex plane is  $\sqrt{e}\mathbb{R}$  where  $e$  is interpreted as complex unit vector in its direction.

**4.2. Preholomorphicity of the observable.** Let  $\lambda = e^{-i\frac{\pi}{4}}$ . Associate to each edge  $e$  of the modified medial lattice one of the following four lines through the origin  $\mathbb{R}, i\mathbb{R}, \lambda\mathbb{R}, \bar{\lambda}\mathbb{R}$  as in Figure 11. Denote this line by  $l(e)$ .

**Spin preholomorphicity:** Choose the constant  $\theta$  in the definition of  $f_\delta$  so that the value of  $f_\delta$  at the edge  $e$  belongs to the line  $l(e)$ . Then  $\theta \in \{\pm 1, \pm i, \pm \lambda, \pm \bar{\lambda}\}$ . The  $\pm$  sign mostly doesn't play any role, but in some situations it should be chosen consistently. The observable  $f_\delta$  satisfies the relation

$$f_\delta(e_W) + f_\delta(e_E) = f_\delta(e_S) + f_\delta(e_N) \quad (25)$$

for every vertex  $v$  of the medial graph, whose four neighboring edges in counter-clockwise order are called  $e_N, e_W, e_S, e_E$ . The relation is verified using the same involution among the loop configurations as in [18].

Therefore, using (25), we can define  $f_\delta(v) := f_\delta(e_W) + f_\delta(e_E) = f_\delta(e_S) + f_\delta(e_N)$  and it satisfies for any neighboring vertices  $v, w$  the identity

$$\text{Proj}_e(f_\delta(v)) = \text{Proj}_e(f_\delta(w)) \quad (26)$$

where  $e$  is the edge between  $v$  and  $w$  and  $\text{Proj}_e$  is the orthogonal projection to the line  $l(e)$ . That is, if  $l(e) = \eta_e\mathbb{R}$  where  $\eta_e$  is a complex number with unit modulus, then

$$\text{Proj}_e(z) = \text{Re}[z\eta_e^*]\eta_e = \frac{z + z^*\eta_e^2}{2}. \quad (27)$$

Since  $f_\delta$  on  $V(G_\delta^\diamond)$  satisfies the relation (26), we call it *spin-preholomorphic* (or *strongly preholomorphic*). Spin-preholomorphic functions satisfy a discrete version of the Cauchy–Riemann equations [18].

**4.3. Martingale property of the observable.** Let  $\gamma = T_{a,d}$  where  $T_{a,d}$  is the branch of the exploration tree constructed in Section 2.3. Let's parametrize  $\gamma$  by lattice steps so that at integer values of the time,  $\gamma$  is at the head of an oriented edge (in  $\mathbb{L}_{\rightarrow}^\spadesuit$ ) between black and white octagon and at half-integer values of the time, it is at the tail of such an edge. Note that step between the times  $t = k - 1/2$  and  $t = k$ ,  $k = 1, 2, \dots$ , is deterministic given the information up to time  $t = k - 1/2$ . Hence between two consecutive integer times, at most one bit of information is generated: namely, whether the curve turn left or right in an “intersection”. Even this choice might be predetermined, if we are visiting a boundary vertex or a vertex visited already before.

Denote the loop configuration by  $\omega$ , in the four marked point setting it consist of two arcs and a number of loops. We note that the pair  $(\gamma[0, t], \omega)$  can be sampled in two different ways:

- (1) In the first option, we sample first  $\omega$  and then  $\gamma[0, t]$  is a deterministic function of  $\omega$  as explained above.

- (2) In the second option, we sample first  $\gamma[0, t]$  (or rather we keep the sample of  $\gamma[0, t]$  of the previous construction) and then we (re-) sample  $\omega'$  in two steps: we sample  $\omega''$  in the complement of  $\gamma[0, t]$  using the boundary condition given by  $\gamma[0, t]$  and then for each visit of  $\gamma[0, t]$  to the arc  $bc \subset V_{\partial,1}$  we flip an independent coin  $\zeta_v \in \{0, 1\}$ ,  $v \in bc$ , such that

$$\mathbb{P}(\zeta_v = 0) = \frac{1}{1 + \sqrt{2}}, \quad \mathbb{P}(\zeta_v = 1) = \frac{\sqrt{2}}{1 + \sqrt{2}} \quad (28)$$

and then we open for each visited  $v \in bc$  such that  $\zeta_v = 1$  the edge  $e_v \in E(\mathbb{L}^\bullet)$  in  $\omega'' \cup \gamma[0, t]$  and call the resulting configuration  $\omega'$ .

See Table 1 for details about the equivalence of these two constructions of  $(\gamma[0, t], \omega)$ . Notice that for  $\gamma$  the column on the right occurs surely, in the sense that the loops of  $\omega$  are merged at boundary points  $v$  until  $\gamma$  reaches  $d$ . The relative weights of the  $\mathbb{L}^\bullet$ -edge at  $v$  being open or closed are calculated in the table giving (28).

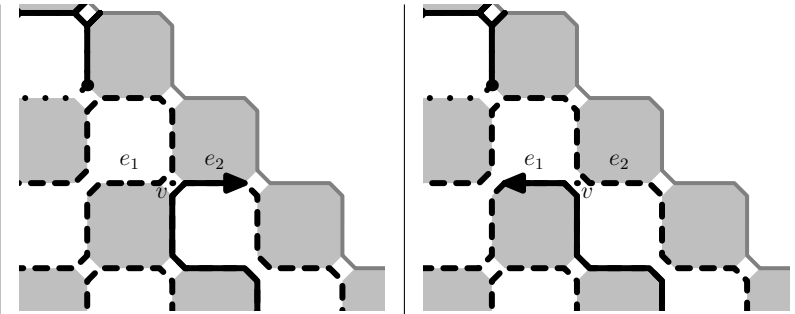
		
weight of configuration with $e_1 \in \hat{\gamma}$	$X\sqrt{2}$	$X$
weight of configuration with $e_1 \notin \hat{\gamma}$	$Y\sqrt{2}$	$Y$
winding at $e$ given $e \in \hat{\gamma}$	$W$	$W \pmod{4\pi}$

TABLE 1. The effect of a jump at a vertex  $v$  to the weight of a loop configuration and the winding of  $\hat{\gamma}$ . The involution preserves the winding factor in the observable and the relative weight of the configuration.

For a non-negative integer  $t$ , let  $G_t^\spadesuit$  be the slitted graph where we have removed  $\gamma[0, t]$  and let  $a_t$  be the edge  $\gamma([t - 1/2, t])$ .

Next we decorate the boundary arc  $bc \subset V_{\partial,1}(G^\diamond)$  with i.i.d. random variables  $(\xi_v)$ ,  $\xi_v \in \{-1, 1\}$  such that

$$\mathbb{P}(\xi_v = -1) = 1 - \frac{1}{\sqrt{2}}, \quad \mathbb{P}(\xi_v = 1) = \frac{1}{\sqrt{2}} \quad (29)$$

Let  $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$  be the filtration such that the  $\sigma$ -algebra  $\mathcal{F}_t$  is generated by  $\gamma[0, t]$  and  $\xi_v$  for any  $v \in \gamma[0, t - 1] \cap bc$ .

Notice that we can couple  $\zeta_v$  and  $\xi_v$  in such a way that

$$\mathbb{P}(\xi_v = 1 \mid \zeta_v = 0) = 1, \quad \mathbb{P}(\xi_v = 1 \mid \zeta_v = 1) = \mathbb{P}(\xi_v = -1 \mid \zeta_v = 1) = \frac{1}{2}. \quad (30)$$

That is, since  $\zeta_v = 1$  is interpreted so that we jump at  $v$  from one boundary touching loop to the neighboring one,  $\xi_v$  is surely equal to 1, if we don't jump at  $v$ , and it is an independent fair coin flip, if we do jump at  $v$ .

Set for any integer  $t$

$$M_t^+ = \mathbb{P}^{G_t^\spadesuit, a_t, b_t, c, d}(\gamma_1 \subset \hat{\gamma}) \quad (31)$$

$$M_t = \left( \prod_{v \in \gamma[0, t-1] \cap bc} \xi_v \right) M_t^+ \quad (32)$$

where we can either take  $b_t = b$  or  $b_t$  is the “rightmost point visible” from  $cd$ .

**Lemma 4.1.**  $(M_t)$  is  $(P, \mathcal{F}_t)$  martingale.

*Proof.* First notice that if  $\gamma(t) \notin bc$ , then  $\mathbb{E}[M_{t+1}^+ | \mathcal{F}_t] = M_t^+$  by the Markov property of random cluster model and hence

$$\mathbb{E}[M_{t+1} | \mathcal{F}_t] = \left( \prod_{v \in \gamma[0, t-1] \cap bc} \xi_v \right) \mathbb{E}[M_{t+1}^+ | \mathcal{F}_t] = M_t. \quad (33)$$

On the other hand, if  $\gamma(t) \in bc$ , then  $M_{t+1}^+ = (1 + \sqrt{2}) M_t^+$  by the calculation of Table 1 and hence

$$\mathbb{E}[M_{t+1} | \mathcal{F}_t] = \left( \prod_{v \in \gamma[0, t-1] \cap bc} \xi_v \right) M_t^+ \underbrace{(1 + \sqrt{2}) \mathbb{E}[\xi_{\gamma(t)} | \mathcal{F}_t]}_{=1} = M_t. \quad (34)$$

Therefore  $M_t$  is a martingale.  $\square$

Define

$$N_t = \theta_\delta \mathbb{E}^{G_t^\spadesuit, a_t, b_t, c, d}(\mathbb{1}_{e \in \hat{\gamma}} e^{-i \frac{1}{2} W(d, e)}) \quad (35)$$

where  $\theta_\delta$  is the constant with unit modulus chosen in Section 4.2. In words, the value of the process  $(N_t)$  at time  $t$  is the value at the given edge  $e$  of the observable on the slit domain at time  $t$ . For the next property  $\theta_\delta$  needs to be chosen consistently. This can be done for example by requiring that the sign of the observable is always the same at  $d$ .

**Lemma 4.2.**  $(N_t)$  is  $(P, \mathcal{F}_t)$  martingale.

*Proof.* If  $\gamma(t) \notin bc$ , then clearly  $\mathbb{E}[N_{t+1} | \mathcal{F}_t] = N_t$ .

If  $\gamma(t) \in bc$ , then by the observation in Table 1,  $N_{t+1} = N_t$ . Therefore  $N_t$  is a martingale.  $\square$

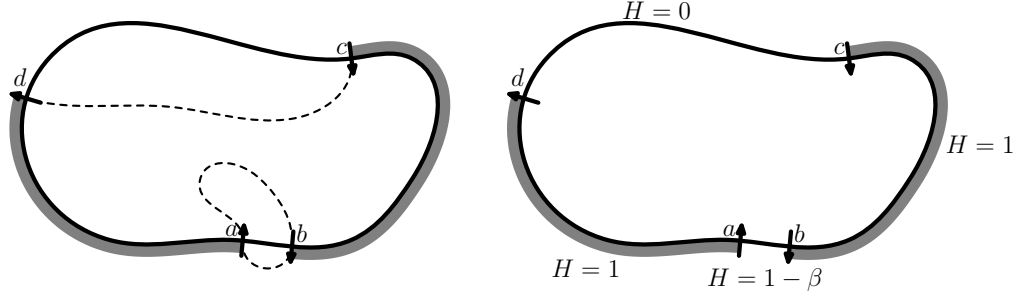
**4.4. Convergence of the observables.** In this subsection we first work on the scaling limit of the *4-point observable*. In that approach we keep the points  $c_\delta$  and  $d_\delta$  at a macroscopically positive distance. That could be considered as a motivation for the so called *3-point* or *fused observable* where we take the same limit, but keeping  $c_\delta$  and  $d_\delta$  at a microscopically bounded distance.

**4.4.1. Convergence of the 4-point observable.** It is straightforward to apply the reasoning of [18] to this case. We only summarize the method here without any proof. See also the lecture notes [8].

The observable  $f_\delta$  is given on the medial lattice. It is preholomorphic on the vertices of the lattice and it has well-defined projections to the complex lines  $l(e)$  defined on the edges of the lattice. Define a function  $H_\delta$  on the square lattice by setting  $H_\delta(B_0) = 1$  where  $B_0$  is the black square next to  $d$  and then extending to other squares by

$$H_\delta(B) - H_\delta(W) = |f_\delta(e)|^2 \quad (36)$$





(a) The setup:  $a$  and  $c$  are inward pointing edges,  $b$  and  $d$  are outward pointing edges. The picture indicates also the boundary conditions:  $ab$  is free,  $bc$  wired etc. We need the external connection from  $b$  to  $a$  to ensure that the random cluster measure is consistent with the one introduced in Section 2.

(b) The discrete function  $H_\delta$  is almost discrete harmonic: its restriction to black squares is discrete subharmonic and to white squares discrete superharmonic. The boundary conditions it satisfies are given in the picture. Notice that there is an unknown constant  $0 < \beta < 1$  which has the interpretation that  $\sqrt{\beta}$  is the probability of an internal connection pattern.

FIGURE 12. The setup for the random cluster measure, for the curves and loops and the boundary conditions satisfied by the (discrete) harmonic functions.

where  $B$  and  $W$  are any pair of neighboring black and white square and  $e$  is the edge of the medial lattice between them. Now  $H_\delta$  is well-defined since by the properties of  $f_\delta$  the sum of differences of  $H_\delta$  along a closed loop is zero. The boundary values  $H_\delta$  are the following, see also Figure 12:

- $H_\delta$  is equal to 0 on the arc  $cd$ .
- $H_\delta$  is equal to 1 on the arcs  $da$  and  $bc$ .
- $H_\delta$  is equal to  $1 - \beta_\delta$  on the arc  $ab$ .

The relation between  $f_\delta$  and  $H_\delta$  becomes more clear after a small calculation. Namely, by this calculation for neighboring black squares  $B, B'$  with a vertex  $v \in L_\diamond$

$$H_\delta(B') - H_\delta(B) = \frac{1}{2} \operatorname{Im} \left( f_\delta(v)^2 \frac{B' - B}{\delta} \right). \quad (37)$$

Notice that the complex number  $(B' - B)/\delta$  has modulus  $\sqrt{2}$ . The natural interpretation is that  $H_\delta(B)$  is the imaginary part of the discrete integral (counted in lattice-step units) of  $\frac{1}{2}f_\delta^2$  from  $B_0$  to  $B$  along any connected path of black squares. Written the lattice-mesh units this means that

$$H_\delta(B') = H_\delta(B) + \operatorname{Im} \int_B^{B'} \left( \frac{1}{\sqrt{2}\delta} f_\delta \right)^2 \quad (38)$$

where the discrete integral is over any lattice path connecting  $B$  to  $B'$  and is defined as the sum over the edges of the path, of the integrand evaluated at the mid point of the edge times the complex number which is the difference of the head and the tail of the edge.

Denote the restriction of  $H_\delta$  to black and white squares by  $H_\delta^\bullet$  and  $H_\delta^\circ$ , respectively. Again by the properties of  $f_\delta$ ,  $H_\delta^\bullet$  is subharmonic and  $H_\delta^\circ$  is superharmonic, see [18]. Let  $\tilde{H}_\delta^\bullet$  be the preharmonic function on the black squares with the same boundary values as  $H_\delta^\bullet$  and similarly  $\tilde{H}_\delta^\circ$  be the preharmonic extension of the boundary values of  $H_\delta^\circ$ . Also extend all these function to be continuous functions, say, by

using the bilinear extension which in takes the form

$$h(x, y) = a_1 + a_2x + a_3y + a_4xy$$

in each square and matches with the values given at the corners of the square. Then at each interior point

$$\tilde{H}_\delta^\circ(z) \leq H_\delta^\circ(z) \leq H_\delta^\bullet(z) \leq \tilde{H}_\delta^\bullet(z). \quad (39)$$

Next apply standard difference estimates to show that the preharmonic functions  $\tilde{H}_\delta$  and  $\tilde{H}_\delta^\bullet$  have convergent subsequences and also using crossing estimates show that their boundary values approach to each other. Since  $0 \leq \beta_\delta \leq 1$ , by taking a subsequence we can show that  $\beta_\delta$  converges to a number  $\beta$  and  $H_\delta$  converges to a harmonic function on  $\Omega$  with the boundary values  $H = 0$  on  $cd$ ,  $H = 1$  on  $bc$  and  $da$  and  $H = 1 - \beta$  on  $ab$ .

As explained in [18] we can extend the convergence of  $H_\delta$  to the convergence of  $f_\delta$  and hence along the same subsequence as  $H_\delta$  converges to  $H$  also  $\frac{1}{\sqrt{2\delta}}f_\delta$  converges to  $f$  defined by

$$f(z) = \sqrt{\phi'(z)} \quad (40)$$

where  $\phi$  is any holomorphic function with  $\text{Im } \phi = H$ .

In fact, the value of  $\beta$  is determined uniquely and it depends only on the conformal type of the domain. We'll give the argument in Section 4.5 for completeness following the lines of Section 4.4.2. Suppose for now that it is the case. Then it follows that the whole sequence  $\frac{1}{\sqrt{2\delta}}f_\delta$  converges.

**4.4.2. A proof for the fused case.** Consider now the same setup, but when  $c$  and  $d$  are close to each other, say, at most at bounded lattice distance from each other. Denote by  $z_0$  the point in the continuum that  $c$  and  $d$  are approximating. We will deal with the case of flat boundary near  $z_0$ .

We will change the definition of  $H$  to that of  $1 - H$ , it means that on the arc  $ab$ ,  $H \equiv \beta$ , on the arcs  $bc$  and  $da$ ,  $H \equiv 0$  and on the arc  $cd$ ,  $H \equiv 1$ . Then  $H$  is superharmonic when restricted to  $\mathbb{L}^\bullet$  and subharmonic when restricted to  $\mathbb{L}^\circ$  and the inequalities in (39) are reversed.

We expect that in the fused (3-point) limit,  $\beta_\delta$  goes to zero, which we have to compensate by renormalizing the observable. In effect, the value of  $H$  on  $cd$  will go to infinity and we expect to get a Poisson kernel type singularity. Hence we say that there is a singularity at  $z_0$  (or at  $c$  or  $d$ ).

We will make the following definitions:

- The discrete half-plane  $\mathbb{H}_\delta^{(z_0, \theta)}$  is a discrete approximation of  $\mathbb{H}^{(z_0, \theta)} := z_0 + e^{i\theta}\mathbb{H}$  where  $\mathbb{H} = \{z \in \mathbb{C} : \text{Im}[z] > 0\}$ . Suppose that the boundary lies between two parallel lines that are at a distance which remains bounded (in lattice steps) as  $\delta \rightarrow 0$ . Assume also that the projection of a parametrization of the boundary on one of the lines is a monotone function, at least in sufficiently large neighborhood of  $z_0$ . This ensures that there are no long fjords near  $z_0$ .
- $H_\delta^+$ ,  $f_\delta^+$  are the half plane functions on  $\mathbb{H}_\delta^{(z_0, \theta)}$  with the “singularity” at  $z_0$ .
- Suppose that  $\Omega_\delta$  is a discrete domain and  $z_0$  is its boundary point. Assume that the boundary in a  $r$  neighborhood of  $z_0$  is flat in the same uniform manner in  $\delta$  as in the definition of the half plane  $\mathbb{H}_\delta^{(z_0, \theta)}$ .
- $H_\delta$ ,  $f_\delta$  are the functions on  $\Omega_\delta$  with the “singularity” at  $z_0$ .
- Suppose that  $\Omega_\delta^0$  that is the domain which is obtained from  $\Omega_\delta$  by changing the boundary conditions on the arc  $ab$  from free to wired.

- $H_\delta^0, f_\delta^0$  are the functions on  $\Omega_\delta^0$  with the “singularity” at  $z_0$ . It has boundary value  $H \equiv 0$  on  $ab$ , too.

The next lemma gives the convergence of the observable in a half-plane. We will compare the other observables to this one.

**Lemma 4.3.** *Let  $w_\delta$  be the discrete approximation of the point  $z_0 + i e^{i\theta}$ . Let  $L_\delta = H_\delta^+(w_\delta)$  and  $\hat{H}_\delta^+ = \frac{1}{L_\delta} H_\delta^+$ . The following statements hold:*

- (i) *As  $\delta \rightarrow 0$  the sequence  $\hat{H}_\delta^+$  converges uniformly on compact sets to  $\text{Im}(-\frac{e^{i\theta}}{z-z_0})$ .*
- (ii) *For each sequence  $\delta_n \searrow 0$  as  $n \rightarrow \infty$ , there exists a subsequence  $\delta_{n_i}$  and a constant  $c^+ \in \mathbb{R}_{>0}$  such that as  $i \rightarrow \infty$  the sequence  $L_{\delta_{n_i}}/\delta_{n_i}$  converges to  $c^+$  and the sequence  $\delta_{n_i}^{-1} H_{\delta_{n_i}}^+$  converges uniformly on compact sets to  $c^+ \text{Im}(-\frac{e^{i\theta}}{z-z_0})$ .*

*Proof.* Let's first prove (i) and then use that result to prove (ii).

We need here that  $\hat{H}_\delta^+(w_\delta) = 1$  and  $\hat{H}_\delta^+ \geq 0$ . Harnack's inequality and Harnack boundary principle [7] imply that for any  $r > 0$ ,  $\hat{H}_\delta^+ \leq C$  in  $\mathbb{H}_\delta^{(z_0, \theta)} \setminus B(z_0, r)$ .

Fix a compact subset of  $\mathbb{H}^{(z_0, \theta)}$  with non-empty interior. From boundedness of  $\hat{H}_\delta^+$  it follows that the corresponding  $f$  function  $\frac{1}{\sqrt{\delta}} \hat{f}_\delta^+$  remains bounded, see [8], and that on the boundary the values on  $\mathbb{L}^\bullet$  and  $\mathbb{L}^\circ$  are close and hence the harmonic extensions of the two functions to the interior are close. Hence along a subsequence  $\hat{H}_\delta^+$  converges to a function on that compact set and the limit is harmonic in the interior points.

By taking an increasing sequence of compact sets we see that the convergence takes place in the whole half-plane for a subsequence. The limit has to be the Poisson kernel of the half-plane normalized to have value 1 at the point  $z_0 + i e^{i\theta}$ . Since the limit is the same for all subsequences, the whole sequence converges.

For (ii) we use the first claim, (i), and the fact that  $H_\delta^+ = L_\delta \hat{H}_\delta^+$ . Notice that the harmonic upper and lower bound for the restrictions of  $H_\delta^+$  to  $\mathbb{L}^\circ$  and  $\mathbb{L}^\bullet$  can be bounded from above and from below, respectively, by a quantity of the form  $\text{const} \cdot \delta$  just by considering the exit probabilities of simple random walks on these two lattices. The best constants that we get might have a gap in between. Nevertheless, there exist constants  $C_1$  and  $C_2$  such that  $0 < C_1 < \delta^{-1} L_\delta < C_2 < \infty$  for small enough  $\delta$ .

Thus we can take a subsequence  $\delta_n$  so that  $\delta_n^{-1} L_{\delta_n}$  converges as  $n \rightarrow \infty$ . Then the claim holds for  $c^+ = \lim_{n \rightarrow \infty} \delta_n^{-1} L_{\delta_n}$ .  $\square$

**Proposition 4.4.** *For each sequence  $\delta_n \searrow 0$  as  $n \rightarrow \infty$  and  $z_0, \theta$  and  $r > 0$ , there exists a subsequence  $\delta_{n_i}$  and constants  $c \in \mathbb{R}_{>0}$  and  $\beta \in \mathbb{R}_{\geq 0}$  such that for any sequence  $\mathbb{H}_\delta^{(z_0, \theta)}$  and for any sequence of domains  $\Omega_\delta$  that agrees with  $\mathbb{H}_\delta^{(z_0, \theta)}$  in the  $r$  neighborhood of  $z_0$  and that converges to a domain  $\Omega$  in the Carathéodory sense, the sequence  $\delta_{n_i}^{-1} H_{\delta_{n_i}}$  converges uniformly on compact sets to a harmonic function  $h$  that satisfies the following boundary conditions*

$$h = \begin{cases} c\beta & \text{on } (a b) \\ 0 & \text{on } (b z_0) \text{ and } (z_0 a) \\ c \text{Im}(-\frac{e^{i\theta}}{z-z_0}) + \mathcal{O}(1) & \text{as } z \rightarrow z_0 \end{cases} \quad (41)$$

Furthermore,  $\delta_{n_i}^{-1} H_{\delta_{n_i}}^+$  converges uniformly on compact sets to  $c^+ \text{Im}(-\frac{e^{i\theta}}{z-z_0})$  along the same subsequence and  $c = c^+$  and the convergence is uniform over all domains  $\Omega$ .

*Proof.* Use the same argument as in the proof of Lemma 4.3 (ii) to show that  $\delta_{n_i}^{-1} H_{\delta_{n_i}}$  converges uniformly on compact sets to a harmonic function  $h$  and it satisfies (41). Suppose that  $c \neq c_+$ . Then

$$\tilde{h} = \text{Im} \int (f - f^+)^2 = (\sqrt{c} - \sqrt{c^+})^2 \frac{\text{Im}((z - z_0) e^{-i\theta})}{|z - z_0|^2} + \mathcal{O}(\text{Im}((z - z_0) e^{-i\theta})) \quad (42)$$

as  $z \rightarrow \infty$ . There exists  $r > 0$  such that  $\tilde{h}$  is positive in  $\mathbb{H}^{(z_0, \theta)} \cap B(z_0, r)$ .

Next we notice that  $\tilde{h}$  is not bounded in  $\mathbb{H}^{(z_0, \theta)} \cap B(z_0, r)$ . On the other hand, if we consider the discrete version  $\tilde{h}_\delta$  of  $\tilde{h}$ , it must remain bounded on  $\mathbb{H}_\delta^{(z_0, \theta)} \cap \partial B(z_0, r)$  uniformly in  $\delta$  because of the convergence to  $\tilde{h}$ . Thus it is bounded inside  $\mathbb{H}_\delta^{(z_0, \theta)} \cap B(z_0, r)$  because of its 0 boundary values on the straight part of the boundary around  $z_0$ . This leads to a contradiction.  $\square$

*Remark 4.5.* If we normalize by the value of  $H^+$ , the convergence holds for the whole sequence, not just along subsequences.

*Remark 4.6.* This proof can be generalized to any domain with  $z_0$  lying on a smooth boundary segment, if the domain is approximated in a nice way around  $z_0$ . This means that the boundary near  $z_0$  lies between two copies of the same smooth curve shifted by a bounded number of lattice steps, for instance. All  $H$  functions on domains that have a given  $r$  neighborhood of  $z_0$  must have the same singular part. Moreover, the value at a fixed point of any fixed  $H$  function can be used for normalization for the other functions and they converge to a limit.

The generalization to non-smooth boundary would require a matching pair of upper and lower bounds for  $H_\delta$ , the former for the restriction to  $\mathbb{L}^\circ$  and the latter to  $\mathbb{L}^\bullet$ . This is therefore equivalent of knowing (uniform bounds for) the leading term of the asymptotics of the exit probabilities with different lattice “boundary shapes” as  $\delta \rightarrow 0$ . The asymptotics is heavily influenced by the local geometry around  $z_0$ .

In the final result of this subsection, we derive a characterizing property for the constant  $\beta$ . Hence this constant is uniquely determined by the continuum setting and doesn't depend on the discrete approximations we are using. In fact, it only depends on the conformal type of the domain  $(\Omega, a, b, z_0)$  with a prescribed length scale (derivative) at  $z_0$ .

Suppose for simplicity that  $\Omega$  is a Jordan domain. Let  $h : \Omega \rightarrow \mathbb{R}$  be a harmonic function, for instance,  $h$  is the scaling limit from Proposition 4.4. Suppose that  $z$  is a boundary point and  $h = \beta$  near  $z$  on the boundary. We define a weak version of the sign of the normal derivative  $\partial_n h(z)$  to the direction of the outer normal at a boundary point  $z$  of a (possibly non-smooth) domain by

- $\partial_n h \geq 0$  at  $w \in ab$ , if

$$h^{-1}((-\infty, \beta]) \cap B(w, \varepsilon) \neq \emptyset$$

for all  $\varepsilon > 0$ .

- $\partial_n h \leq 0$  at  $w \in bc \cup ca$ , if

$$h^{-1}([\beta, \infty)) \cap B(w, \varepsilon) \neq \emptyset$$

for all  $\varepsilon > 0$ .

**Proposition 4.7.** *The constant  $\beta$  in (41) is the unique constant such that  $h$  has the normal derivatives*

$$\begin{cases} \partial_n h \geq 0 & \text{on } ab \\ \partial_n h \leq 0 & \text{on } bc \text{ and } ca \end{cases} \quad (43)$$

and there exists a point  $w$  on  $ab$  such that  $h^{-1}((\beta, \infty)) \cap B(w, \varepsilon) \neq \emptyset$  for all  $\varepsilon > 0$ .

*Proof.* The normal derivatives (43) follow by the same argument as in [7], Section 6.

Suppose that  $\phi : U \rightarrow \mathbb{H}$  is conformal and onto and  $\phi(c) = \infty$ . Then  $h = h^{\mathbb{H}} \circ \phi$  where

$$h^{\mathbb{H}}(z) = \text{Im} \int (f^{\mathbb{H}})^2 \quad (44)$$

and up to a universal multiplicative constant

$$f^{\mathbb{H}}(z) = \sqrt{1 + \beta \left( -\frac{1}{z-u} + \frac{1}{z-v} \right)}, \quad (45)$$

where  $u = \phi(a)$  and  $v = \phi(b)$ .

Write

$$f^{\mathbb{H}}(z) = \sqrt{\frac{Q(z)}{(z-u)(z-v)}}, \quad (46)$$

where

$$Q(z) = z^2 - (u+v)z + (uv + \beta(v-u)). \quad (47)$$

Since the coefficients of the quadratic polynomial  $Q$  are real, it either has two real zeros or a pair of complex conjugate zeros (with non-zero imaginary part). Since (46) holds and  $f^{\mathbb{H}}$  is single valued in  $\mathbb{H}$ , there can't be any zeros of multiplicity one in the upper half-plane. Thus the zeros of  $Q$  are real.

Now

$$\partial_n h^{\mathbb{H}}(x) = -\partial_y h^{\mathbb{H}}(x) = -\text{Re}[(f^{\mathbb{H}}(x))^2] = \frac{Q(x)}{(x-u)(v-x)} \quad (48)$$

on  $\mathbb{R} \setminus \{u, v\}$ . Let the zeros of  $Q$  be  $w_1$  and  $w_2$  with  $w_1 \leq w_2$ . Then  $w_1 + w_2 = u + v$  by (47). Therefore we have to analyze the following four cases

- (1)  $u < w_1 = w_2 < v$
- (2)  $u < w_1 < w_2 < v$
- (3)  $u = w_1 < w_2 = v$
- (4)  $w_1 < u < v < w_2$ .

We notice using (48) that only the first case is consistent with (43).

Thus we have shown that  $w_1 = w_2 = (u+v)/2$  and hence  $\beta = (v-u)/4$ . The normal derivative  $\partial_n h^{\mathbb{H}}(x)$  is positive when  $x \in (u, v)$  and negative when  $x \in \mathbb{R} \setminus [u, v]$ . The value of  $\beta$  is uniquely determined and it is the only real value such that  $\partial_n h$  has the properties claimed.  $\square$

**4.5. Value of  $\beta$  for the 4-point observable.** We will determine the value of  $\beta$  for the 4-point observable in the same way as in Proposition 4.7. Remember that  $0 \leq \beta \leq 1$ .

Consider  $f^{\mathbb{H}}$  for the domain  $\mathbb{H}$  with the four marked boundary points  $u < v < w$  and  $\infty$ . This means that  $h$  can be written as  $h = h^{\mathbb{H}} \circ \phi$  where

$$h^{\mathbb{H}} = \text{Im} \int (f^{\mathbb{H}})^2 \quad (49)$$

and  $\phi(a) = u$ ,  $\phi(b) = v$ ,  $\phi(c) = w$  and  $\phi(d) = \infty$ .

For any  $u < v < w$

$$h^{\mathbb{H}, u, v, w}(z) = \frac{1}{\pi} \text{Im} (-\log(z-w) + \beta(-\log(z-u) + \log(z-v))). \quad (50)$$

Hence

$$\begin{aligned}\sqrt{\pi} f^{\mathbb{H}, u, v, w}(z) &= \sqrt{-\frac{1}{z-w} + \beta \left( -\frac{1}{z-u} + \frac{1}{z-v} \right)} \\ &= \sqrt{-\frac{Q(z)}{(z-u)(z-v)(z-w)}}\end{aligned}\quad (51)$$

where  $Q(z)$  is a quadratic polynomial.

Let's simplify things by setting  $u = 0$  and  $w = 1$ . Hence for  $0 < v < 1$ ,  $Q(z)$  can be written as

$$\begin{aligned}Q(z) &= z(z-v) - \beta v(z-1) \\ &= z^2 - (\beta+1)vz + \beta v.\end{aligned}\quad (52)$$

Since the coefficients of  $Q$  are real, there are two options: *either* we have that  $\text{Im } w_1 \neq 0$  or  $\text{Im } w_2 \neq 0$ , and then  $w_1^* = w_2$ , *or* we have that  $w_1$  and  $w_2$  are real. Since (51) holds and  $f^{\mathbb{H}}$  is single valued in  $\mathbb{H}$ , there can't be any zeros in the upper half-plane. Hence the zeros are real.

Let's write also in this case the normal derivative in the direction of the outer normal

$$\partial_n h^{\mathbb{H}}(x) = -\partial_y \text{Im} \int (f^{\mathbb{H}})^2 = \text{Re} \frac{Q(x)}{x(x-v)(x-1)} \quad (53)$$

for all  $x \in \mathbb{R} \setminus \{0, v, 1\}$ . By the same argument as in [7], Section 6, this normal derivative is negative on  $(-\infty, 0) \cup (v, 1)$  and positive on  $(0, v) \cup (1, \infty)$ . This is only possible if the two roots of  $Q$  are equal. Therefore in addition to  $0 < \beta < 1$ , the constant  $\beta$  has to satisfy

$$(\beta+1)^2 v^2 = 4\beta v \quad (54)$$

and hence  $\beta = \beta_-$  or  $\beta = \beta_+$  where

$$\beta_{\pm} = \frac{-v + 2 \pm 2\sqrt{1-v}}{v}.$$

Let's write  $\beta_{\pm} - 1 = \frac{2}{v} \sqrt{1-v} (\sqrt{1-v} \pm 1)$ , which is positive for  $\beta_+$  and negative for  $\beta_-$  for all  $v \in (0, 1)$ . Thus  $\beta = \beta_-$  and we find that

$$\beta = \frac{-v + 2 - 2\sqrt{1-v}}{v} = \left( \frac{1 - \sqrt{1-v}}{\sqrt{v}} \right)^2 = \left( \tan \left( \frac{x}{2} \right) \right)^2. \quad (55)$$

where  $0 < x < \pi/2$  is such that  $v = \sin^2 x$ . Notice that the double root of  $Q$  is  $\sqrt{\beta}v$  and it lies in the interval  $(0, v)$ .

To conclude, we state that the value of  $\beta$  is characterized uniquely by the formula (50), the fact that  $\beta \in (0, 1)$  and that the normal derivative of  $h^{\mathbb{H}}$  is negative on  $(-\infty, u) \cup (v, w)$  and positive on  $(u, v) \cup (w, \infty)$ .

**4.6. A remark on crossing probabilities.** As a side remark, let's derive the probability  $P$  of the internal arc pattern

$$(0 \frown \infty, 1 \frown v) \quad (56)$$

under the random cluster measure where there isn't any exterior connections, that is, the arcs  $\gamma_1$  and  $\gamma_2$  (as defined in Section 4.1) are not counted as loops in the weight of a configuration. Since  $\sqrt{\beta}$  and  $1 - \sqrt{\beta}$  are proportional to  $P$  and  $\sqrt{2}(1 - P)$ , respectively,  $P$  satisfies

$$\frac{P}{P + \sqrt{2}(1 - P)} = \sqrt{\beta} \quad (57)$$

and hence

$$P = \frac{\sqrt{2}\sqrt{\beta}}{(\sqrt{2}-1)\sqrt{\beta}+1} = \frac{\sin\left(\frac{x}{2}\right)}{\sin\left(\frac{x}{2}\right) + \sin\left(\frac{\pi}{4} - \frac{x}{2}\right)}. \quad (58)$$

By using the relations

$$\sin\left(\frac{x}{2}\right) = \frac{1}{\sqrt{2}}\sqrt{1 - \sqrt{1-v}}, \quad \sin\left(\frac{\pi}{4} - \frac{x}{2}\right) = \frac{1}{\sqrt{2}}\sqrt{1 - \sqrt{v}} \quad (59)$$

we can write this into the form

$$P = \frac{\sqrt{1 - \sqrt{1-v}}}{\sqrt{1 - \sqrt{1-v}} + \sqrt{1 - \sqrt{v}}}. \quad (60)$$

This is consistent with the result in [7].

## 5. CHARACTERIZATION OF THE SCALING LIMIT

### 5.1. Martingales and uniform convergence with respect to the domain.

Consider the scaling limit of a single branch from  $a$  to  $z_0$  in the domain  $\Omega$ . This is described after mapping to a reference domain  $\mathbb{D}$  so that the law  $\mathbb{P}^*$  of a random curve in  $\mathbb{D}$  from  $-1$  to  $1$  is any subsequent scaling limit of  $\mathbb{P}_\delta$ . Consider some scheme of parametrizing the curves that works for all  $\delta$ . We will mostly use the half-plane capacity parametrization. Denote by  $\mathcal{F}_t$  the filtration generated by the curve up to time  $t$ .

After we start exploring the branch we will move automatically from the setting of two points to a setting of three points. Hence we will also consider the setups of  $(\Omega, a, b, z_0)$ , where  $z_0$  is the fused arc  $(cd)$ , and  $(\Omega, a, b, c, d)$ .

Remember that the two martingales were

$$M_t^{(\delta)} = \pm \sqrt{\frac{1}{\delta}\beta_t^{(\delta)}} = \pm \sqrt{\frac{1}{\delta}\beta^{(\delta)}(\Omega_t, a_t, b_t, z_0)} \quad (61)$$

$$N_t^{(\delta)} = \frac{1}{\delta}f_t^{(\delta)}(w_0) = \frac{1}{\delta}f_t^{(\delta), \Omega_t, a_t, b_t, z_0}(w_0) \quad (62)$$

Notice that we have included the scaling by a power of  $\delta$  that makes these quantities converge in the limit  $\delta \rightarrow 0$ , at least for subsequences.

Consider one of the processes above, for instance,  $(M_t^{(\delta)})_{t \geq 0}$  the martingale property could be formulated so that if  $0 \leq s < t$  and if  $\psi : X \rightarrow \mathbb{R}$  is bounded, uniformly continuous and  $\mathcal{F}_t$ -measurable, then

$$\mathbb{E}^{(\delta), \Omega, a, b, z_0}[\psi M_t^{(\delta)}] = \mathbb{E}^{(\delta), \Omega, a, b, z_0}[\psi M_s^{(\delta)}] \quad (63)$$

Now due to the uniform convergence of  $\beta^{(\delta)}(\Omega_t, a_t, b_t, z_0)$  over the domains, Proposition 4.4, the expected values on both sides will converge and we get

$$\mathbb{E}^{*, \Omega, a, b, z_0}[\psi M_t] = \mathbb{E}^{*, \Omega, a, b, z_0}[\psi M_s] \quad (64)$$

where  $\tilde{M}_t = \lim_{n \rightarrow \infty} M_t^{(\delta_n)} = \pm \sqrt{\beta(\Omega_t, a_t, b_t, z_0)}$ . Thus  $(\tilde{M}_t)_{t \geq 0}$  is a martingale.

By a similar argument,  $\tilde{N}_t = \lim_{n \rightarrow \infty} N_t^{(\delta_n)} = f_t^{\Omega_t, a_t, b_t, z_0}(w_0)$  defines a martingale.

**5.2. Simple martingales and a martingale problem.** We wrote  $f$  in the upper half-plane already in (45). Let us now analyze what happens for a growing curve which we interpret as a random Loewner chain. For that we use Theorem 3.11. Next we notice that for all domains (and their approximating sequences) that agree near  $z_0$ , we had singularity in  $H$  with the same constant in front, see Proposition 4.4. Fix some domain  $(\Omega, a, b, z_0)$  and map it to the upper half-plane conformally. Suppose that  $w$  is the image of  $z_0$ . Then the singularity is of the form  $c \operatorname{Im}(-1/(z-w))$ . If we have a slit domain  $(\Omega \setminus \gamma[0, t], \gamma(t), b, z_0)$  and we apply further the Loewner map  $g_t$  in the upper half-plane, then the singularity has to be  $c \operatorname{Im}(-g'_t(w)/(g_t(z) - g_t(w))) = c \operatorname{Im}(-1/(z-w)) + o(1)$ , as  $z \rightarrow w$ . This shows that the functions  $H$  transform as

$$H^{\mathbb{H} \setminus K_t, U_t, V_t, w}(z) = \sqrt{g'_t(w)} H^{\mathbb{H}, g_t(U_t), g_t(V_t), g_t(w)}(g_t(z)). \quad (65)$$

Since  $f = \sqrt{2i\Phi'}$  where  $\Phi$  is holomorphic and  $\operatorname{Im} \Phi = H$ ,

$$f^{\mathbb{H} \setminus K_t, U_t, V_t, w}(z) = \sqrt{g'_t(w) g'_t(z)} f^{\mathbb{H}, g_t(U_t), g_t(V_t), g_t(w)}(g_t(z)). \quad (66)$$

Now if we choose to send  $w$  to  $\infty$ , then the observable is of the form (45). For Loewner chains  $g'_t(\infty) = 1$  when appropriately interpreted, and hence

$$f^{\mathbb{H} \setminus K_t, U_t, V_t, \infty}(z) = \sqrt{g'_t(z)} \sqrt{1 + \beta_t \left( -\frac{1}{g_t(z) - U_t} + \frac{1}{g_t(z) - V_t} \right)} \quad (67)$$

As we saw in the proof of Proposition 4.7, the value of  $\beta_t$  is

$$\beta_t = \frac{1}{4}(V_t - U_t).$$

We define

$$M_t = \pm \sqrt{4\beta_t} = \pm \sqrt{V_t - U_t} \quad (68)$$

$$N_t = 4(\beta_t(V_t - U_t) - 2t) = M_t^4 - 8t. \quad (69)$$

The former quantity is proportional to  $\tilde{M}_t$  and the latter one to the first non-trivial coefficient in the expansion of (67) around  $z = \infty$ .

By the martingale properties in Section 4.1 and the convergence results of the observables we have the following result.

**Proposition 5.1.** *Let  $\mathbb{P}^*$  be a subsequent limit of the sequence of laws of FK Ising branch in discrete approximations of  $(\Omega, a, b, z_0)$ . Let  $\phi : \Omega \rightarrow \mathbb{H}$  be a conformal, onto map such that  $\phi(z_0) = \infty$ . Let  $\gamma$  be the random curve distributed according to  $\mathbb{P}^*$  in the capacity parametrization,  $U_t = \phi(\gamma(t))$  and  $V_t$  is the “right-most point” in the hull of  $\phi(\gamma(t))$ . Let the signs in (68) be i.i.d. fair coin flips independent of  $\gamma$ . Then processes  $(M_t)_{t \geq 0}$  and  $(N_t)_{t \geq 0}$  are martingales.*

In the rest of this section we will show that the following *martingale problem* as a unique solution:

Let  $(U_t, V_t)_{t \geq 0}$ ,  $(M_t)_{t \geq 0}$  and  $(N_t)_{t \geq 0}$  as above. What is their law given that  $(M_t)_{t \geq 0}$  and  $(N_t)_{t \geq 0}$  are martingales?

The solution is divided into two part. In Section 5.3 we will show that  $(|M_t|^{\frac{1}{\alpha}})_{t \geq 0}$  for some  $\alpha > 0$  is a Bessel process. In Section 5.4 we will show that  $(V_t)_{t \geq 0}$  follows the evolution based on the Loewner equation and there won't be any singular process in  $(V_t)_{t \geq 0}$  which would “live” on the random Cantor set  $\{t : U_t = V_t\}$ .

**5.3. Characterization of  $V_t - U_t$ .** In this section, we show how the “martingale problem” characterizes the law of  $(V_t - U_t)$ .



5.3.1. *Relation to Lévy's and Stroock–Varadhan martingale characterizations.* The argument which we will present can be compared to Paul Lévy's characterization of Brownian motion. The law of Brownian motion  $(B_t)_{t \geq 0}$  is characterized by the fact that  $B_t$  and  $B_t^2 - t$  are martingales. In the setting of general diffusions, the classical Stroock–Varadhan martingale problem approach describes weak solutions  $X_t$  to stochastic differential equations of the form  $dX_t = \sqrt{a} dB_t + b dt$  (with coefficients  $a$  and  $b$  satisfying suitable measurability conditions) as exactly those that all the quantities of the form  $f(X_t) - \int_0^t (\frac{1}{2} a_s(X) f''(X_s) + b_s(X) f'(X_s)) ds$  are martingales for a class of test functions  $f$ , see Sections V.19 and V.20 in [15]. However, similarly to Lévy's theorem, there are stronger results, stating that two (well-chosen) martingales are enough to characterize the law of a diffusion, see [2] and [19]. We take this path, using two martingales to show that the diffusion in question is the Bessel process.

5.3.2. *Lemmas.* We need the next two lemmas, which we write in greater generality suitable for the 4-point case.

**Lemma 5.2.** *Suppose that  $T > 0$  is a stopping time and suppose that  $\psi \in C^2$  satisfies  $\psi''(m) > 0$  when  $m \neq 0$ ,  $\psi(0) = 0$  and  $\psi''(0) = 0$ . If  $A_t$  and  $C_t$  are continuous, predictable processes which satisfy*

- (1)  $A_t$  is non-increasing and positive
- (2)  $C_t$  is non-decreasing and differentiable and  $\dot{C}_t > 0$  almost surely on  $[0, T)$ .

*then any continuous martingale  $(M_t)_{t \in [0, T]}$  with the property that the process  $(N_t)_{t \in [0, T]}$  defined by*

$$N_t = A_t \psi(M_t) - C_t \quad (70)$$

*is a martingale, satisfies*

$$\mathbb{P} \left[ \int_0^T \mathbb{1}_{M_t=0} dt = 0 \right]. \quad (71)$$

*Proof.* Since  $(M_t)_{t \in [0, T]}$  is a continuous martingale, it has quadratic variation process  $\langle M \rangle_t$ . By Itô's formula

$$dN_t = \psi(M_t) dA_t - \dot{C}_t dt + \frac{1}{2} A_t \psi''(M_t) d\langle M \rangle_t + A_t \psi'(M_t) dM_t. \quad (72)$$

Since  $\psi(M_t)$  and  $\psi''(M_t)$  vanish when  $M_t = 0$ , it follows that

$$\int_0^t \mathbb{1}_{M_s=0} dN_s - \int_0^t \mathbb{1}_{M_s=0} A_s \psi'(M_s) dM_s = - \int_0^t \mathbb{1}_{M_s=0} \dot{C}_s ds. \quad (73)$$

The process  $\mathbb{1}_{M_t=0}$  is predictable, because it is a pointwise limit of measurable functions (for instance,  $\mathbb{1}_{M_t=0} = \lim_{n \rightarrow \infty} \max\{0, 1 - n|M_t|\}$ ), and hence the left-hand side of (73) is a local martingale. On the other hand the right-hand side of (73) is bounded variation process. Hence  $\int_0^t \mathbb{1}_{M_s=0} \dot{C}_s ds = 0$  almost surely. Since  $\dot{C}_s > 0$ , the claim follows.  $\square$

**Lemma 5.3.** *If  $(M_t)_{t \in [0, T]}$  is a continuous martingale with respect to  $(\mathcal{F}_t)_{t \in [0, T]}$  such that*

- (1) *almost surely  $\int_0^T \mathbb{1}_{M_t=0} dt = 0$*
- (2)  *$d\langle M \rangle_t = \sigma_t^2 dt$  and  $\sigma_t > 0$  for any  $t$  such that  $M_t \neq 0$ ,*

*then  $(B_t)_{t \in [0, T]}$  defined as*

$$B_t = \int_0^t \mathbb{1}_{M_s \neq 0} \sigma_s^{-1} dM_s \quad (74)$$

is well-defined and continuous and it is a one-dimensional standard  $(\mathcal{F}_t)$ -Brownian motion.

*Proof.* Let  $F = [0, T] \setminus \{t : M_t = 0\}$  which is relatively open in  $[0, T]$ . For any  $(t_{k-1}, t_k] \subset F$ , it holds that  $\int \mathbb{1}_{(t_{k-1}, t_k]} \sigma_s^{-2} d\langle M \rangle_s = t_k - t_{k-1}$  by the assumptions. We can approximate the set  $F$  from below by finite unions of this type of intervals and use monotone convergence theorem to show that

$$\int_0^t \mathbb{1}_F \sigma_s^{-2} d\langle M \rangle_s = \int_0^t \mathbb{1}_F ds \quad (75)$$

where the left-hand side is a Lebesgue–Stieltjes integral and the right-hand side is a Lebesgue integral defined pointwise in the randomness almost surely.

This shows that  $\mathbb{1}_F \sigma_s^{-1}$  is  $dM_s$  integrable (belongs to the square integrable processes with respect to the variation process of  $M$ ) and  $(B_t)$  is well-defined and continuous in  $t$ . Therefore clearly,  $(B_t)$  is a local martingale and satisfies  $\langle B \rangle_t = t$ . Hence  $(B_t)$  is a standard Brownian motion by Lévy's characterization theorem.  $\square$

5.3.3.  $V_t - U_t$  is a Bessel process. Let  $X_t = V_t - U_t$ . The next theorem will show that  $(X_t)_{t \geq 0}$  is a Bessel process of dimension  $\delta = 3/2$ .

**Theorem 5.4.** *Let  $(a, b) \subset \mathbb{R}$ . Suppose that  $\psi : (a, b) \rightarrow \mathbb{R}$  is twice continuously differentiable function which is convex, i.e.  $\psi'' \geq 0$ , and such that  $\psi'$  is strictly increasing. Let  $M = (M_t)_{t \in \mathbb{R}_+}$  be a continuous stochastic process adapted to a filtration  $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$  and  $N = (N_t)_{t \in \mathbb{R}_+}$  a process defined by  $N_t = 2\psi(M_t) - t$ . Suppose that  $M$  and  $N$  are martingales. Then the following claims hold.*

- (1) *If the process  $W = (W_t)_{t \in \mathbb{R}_+}$  is defined as*

$$W_t = \int_0^t \sqrt{\psi''(M_s)} dM_s,$$

*then it is a standard Brownian motion.*

- (2) *If for some  $\varepsilon > 0$ ,  $\psi(x) = |x|^{2+\varepsilon}$ , then there exists constants  $C > 0$  and  $1 < \delta < 2$  such that  $Z_t = C\psi(M_t)$  is a squared Bessel process of dimension  $\delta$ .*
- (3) *Suppose there exists continuous functions  $F, \tilde{F}$  such that  $2\tilde{F}(x) = F(2\psi(x))$  and  $\psi'(x) = \text{sgn}(\psi'(x))\tilde{F}(x)\sqrt{\psi''(x)}$  for all  $x$  — in particular  $\psi''$  is positive except possibly at the point (there exists at most one such point) where  $\psi'$  is zero. Then  $Z_t = 2\psi(M_t)$  is a solution to the stochastic differential equation*

$$dZ_t = F(Z_t)d\tilde{W}_t + dt \quad (76)$$

*for some standard Brownian motion  $(\tilde{W}_t)_{t \in \mathbb{R}_+}$ .*

*Proof.* (1) Similarly as in the previous section, we notice that we can do stochastic analysis with  $M$ , because  $M$  is a continuous martingale, see Chapter 2 of [9]. The same argument as above tells us that the process defined as

$$W_t = \int_0^t \sqrt{\psi''(M_s)} dM_s,$$

is a continuous martingale with a variation process  $\langle W \rangle_t = t$ . Hence by Lévy's characterization theorem, it is a standard Brownian motion.

- (2) When  $\psi(x) = |x|^{2+\varepsilon}$ , there is a constant  $D > 0$  such that

$$\psi'(x) = D \text{sgn}(x) \sqrt{\psi''(x)\psi(x)}$$

Therefore  $Z_t = 2C\psi(M_t)$  satisfies

$$\begin{aligned} dZ_t &= 2C\psi'(M_t) dM_t + C\psi''(M_t) d\langle M \rangle_t \\ &= 2C D \operatorname{sgn}(M_t) \sqrt{\psi(M_t)} dW_t + C dt \\ &= 2 \sqrt{C/2} D \operatorname{sgn}(M_t) \sqrt{Z_t} dW_t + C dt. \end{aligned}$$

Hence if we choose  $C = 2D^{-2}$ ,  $Z_t$  is squared Bessel process with the parameter  $\delta = 2D^{-2}$ . Here we used the fact that  $\int_0^t \operatorname{sgn}(M_s) dW_s$  is a standard Brownian motion by Lévy's characterization theorem.

A direct calculation shows that  $D = \sqrt{(2+\varepsilon)/(1+\varepsilon)} \in (1, \sqrt{2})$ .

(3) Similarly as above we can write

$$\begin{aligned} dZ_t &= 2\psi'(M_t) dM_t + \psi''(M_t) d\langle M \rangle_t \\ &= 2 \operatorname{sgn}(\psi'(M_t)) \tilde{F}(M_t) dW_t + dt \\ &= \operatorname{sgn}(\psi'(M_t)) F(Z_t) dW_t + dt \end{aligned}$$

from which the claim follows.  $\square$

*Remark 5.5.* Note that when  $1/\alpha = 2+\varepsilon$  and  $\alpha = 1-\delta/2$  and  $D = \sqrt{(2+\varepsilon)/(1+\varepsilon)}$ , then  $2D^{-2} = \delta$ .

**5.4. Characterization of  $(U_t, V_t)$ .** Since  $(X_t)_{t \geq 0}$  is a Bessel process of dimension  $\delta = 3/2$ , we see that  $\int_0^t X_s^{-1} ds$  is finite. Therefore we have shown so far that

$$V_t = V_0 + 2 \int_0^t \frac{ds}{X_s} + \Lambda_t \quad (77)$$

$$U_t = V_0 + 2 \int_0^t \frac{ds}{X_s} - X_t + \Lambda_t \quad (78)$$

where  $\Lambda_t$  is some non-decreasing process which is constant on any subinterval of  $\{t : X_t > 0\}$ . See also Proposition C.2 in Appendix C. In this section we will show that  $\Lambda_t \equiv 0$ .

Let  $\Sigma = \{t : X_t = 0\}$ . The *index* of a Bessel process is defined as

$$\nu = \frac{\delta}{2} - 1. \quad (79)$$

Notice that  $\nu = -1/4$  in our case when the Bessel process has dimension  $3/2$ . By [14] Exercise XI.1.25 and [3] Theorem III.15, the Hausdorff dimension of the support of the local time of a Bessel process with index  $\nu \in [-1, 0]$  is  $-\nu$ . Hence the Hausdorff dimension  $\dim_{\mathcal{H}}(\Sigma) = 1/4$ .

Now  $\Lambda_t \equiv 0$  follows from the next lemma and from the fact that  $(U_t)_{t \in [0, T]}$  is  $1/2 - \varepsilon$  Hölder continuous for any  $\varepsilon > 0$  by Theorem 3.11.

**Lemma 5.6.** *Let  $I \subset [0, S]$  be a closed set with Hausdorff dimension  $\alpha_0 \in [0, 1]$ . Suppose that  $f : [0, S] \rightarrow \mathbb{R}$  is continuous and constant on any subinterval of  $[0, S] \setminus I$ . If  $f$  is not a constant function and if  $f$  is Hölder continuous with exponent  $\alpha$ , then  $\alpha \leq \alpha_0$ .*

The proof is standard using the definition of Hausdorff measure and

$$|f(x) - f(y)| \leq C|x - y|^\alpha,$$

and we leave it to industrious reader.

**5.5. The martingale characterization in the 4-point case.**

5.5.1. *Simple martingales from the observable.* Let  $U_t$  be the driving process,  $V_t$  the point corresponding to  $b_t$ , and  $W_t$  the point corresponding to  $c$  then set

$$X_t = V_t - U_t \quad (80)$$

$$Y_t = W_t - V_t. \quad (81)$$

Then given  $\gamma[0, t]$  in  $\mathbb{H}$ , the map

$$z \mapsto \frac{g_t(z) - U_t}{W_t - U_t} \quad (82)$$

maps  $H_t = \mathbb{H} \setminus \gamma[0, t]$  onto  $\mathbb{H}$  and the marked points to 0,  $X_t/(X_t + Y_t)$  and 1. Therefore

$$\sqrt{\pi} f^{\mathbb{H}, U_t, V_t, W_t}(z) = \sqrt{g'_t(z)} \sqrt{\frac{1}{g_t(z) - W_t} + \beta_t \left( \frac{1}{g_t(z) - U_t} - \frac{1}{g_t(z) - V_t} \right)} \quad (83)$$

where

$$\beta_t = \beta \left( \frac{X_t}{X_t + Y_t} \right) = \left( \sqrt{1 + \frac{Y_t}{X_t}} - \sqrt{\frac{Y_t}{X_t}} \right)^2 \quad (84)$$

The quantity

$$M_t = \pm \sqrt{\beta_t} \quad (85)$$

is a conditional probability of event  $\gamma_1 \subset \hat{\gamma}$  hence  $(M_t)_{t \geq 0}$  is a martingale. Here  $\pm$  sign is needed to extend the martingale property beyond the hitting of 0 by  $M_t$ .

From the expansion of  $f$  as  $z \rightarrow \infty$ , we get that

$$N_t = X_t M_t^2 - W_t \quad (86)$$

is a martingale.

Write (86) as

$$N_t = Y_t \psi(M_t) - W_t. \quad (87)$$

where  $\psi(m) = 4[m^2/(1 - m^2)]^2$ . Note that  $W_t$  is always differentiable and  $Y_t$  is differentiable outside the set of times  $\Sigma := \{t : X_t = 0\} = \{t : M_t = 0\}$ . Now we have to solve the following *martingale problem*:

for given filtered probability space  $(\mathbb{P}^*, (\mathcal{F}_t))$ , determine the law of  $(M_t, Y_t, W_t)_{t \in [0, T]}$  which satisfies

- (1)  $(W_t)$  is strictly increasing and  $C^1$  for all  $t \in [0, T]$
- (2)  $(Y_t)$  is strictly increasing and  $C^1$  for all  $t \in [0, T] \setminus \Sigma$
- (3)  $\dot{W}_t$  is given by

$$\dot{W}_t = \frac{2}{Y_t} \left( \frac{M_t - M_t^{-1}}{M_t + M_t^{-1}} \right)^2 \quad (88)$$

and for a non-decreasing process  $\Lambda_t$  given by Proposition C.2 in Appendix C, it holds that

$$Y_t = Y_0 - \int_0^t \frac{1}{2Y_s} \left( \frac{(M_s - M_s^{-1})^2}{M_s + M_s^{-1}} \right)^2 ds - \Lambda_t. \quad (89)$$

- (4)  $(M_t)$  and  $(N_t)$  (defined by (87)) are martingales.

5.5.2. *Solving the martingale problem.* By the lemmas of Section 5.3.2, we can construct out of  $(M_t)$ , which is a continuous martingale, a Brownian motion  $(B_t)$  so that there exists a process  $(\sigma_t)$ ,  $\sigma_t \geq 0$ , both adapted to  $(\mathcal{F}_t)$ , such that

$$\langle M \rangle_t = \int_0^t \sigma_s^2 ds, \quad M_t = M_0 + \int_0^t \sigma_s dB_s. \quad (90)$$

Hence we know that there exists a  $(\mathbb{P}^*, (\mathcal{F}_t))$  Brownian motion  $(B_t)$  and when  $M_t \neq 0$ ,  $dM_t = \sigma_t dB_t$ . Write the drift of the process  $(N_t)$  using Itô's lemma as

$$\sigma_t^2 \cdot \frac{24 Y_t M_t^2 (M_t^2 + 1)}{(M_t^2 - 1)^4} - \frac{2(M_t^2 - 1)^2}{Y_t (M_t^2 + 1)}. \quad (91)$$

Therefore we have to have

$$\sigma_t = \frac{1}{2\sqrt{3}} \frac{(1 - M_t^2)^3}{Y_t |M_t| (M_t^2 + 1)}. \quad (92)$$

Hence the solution of the martingale problem is that

$$M_t = M_0 + \int_0^t \frac{1}{2\sqrt{3}} \frac{(1 - M_s^2)^3}{Y_s |M_s| (M_s^2 + 1)} dB_s. \quad (93)$$

If we plug this into the expression of  $X_t$ , we get

$$dX_t = \frac{4}{\sqrt{3}} dB_t + \frac{1}{3} \frac{(3M_t^4 + 2M_t^2 + 1)(M_t^2 - 1)^2}{Y_t M_t^2 (M_t^2 + 1)^2} dt. \quad (94)$$

Notice that when  $X_t/Y_t \rightarrow 0$  then

$$M_t \approx \pm \frac{1}{2} \sqrt{\frac{X_t}{Y_t}} \quad (95)$$

and (94) becomes

$$dX_t \approx \frac{4}{\sqrt{3}} dB_t + \frac{4}{3} \frac{dt}{X_t}. \quad (96)$$

which corresponds to the Bessel process with dimension  $\delta = \frac{3}{2}$ , as it should.

In the same way,

$$\sigma_t \approx \frac{1}{\sqrt{3}} \frac{1}{\sqrt{X_t Y_t}} \quad (97)$$

Hence  $\sigma_t^2$  is integrable with respect to  $dt$  in the same sense as  $X_t^{-1}$ .

5.5.3. *Characterization of  $(U_t, V_t, W_t)_{t \in [0, T]}$ .* By comparing to a Bessel process we see that  $\int_0^t X_s^{-1} ds$  is finite. Therefore we have shown that

$$V_t = V_0 + 2 \int_0^t \frac{ds}{X_s} + \Lambda_t \quad (98)$$

$$U_t = V_0 + 2 \int_0^t \frac{ds}{X_s} - X_t + \Lambda_t \quad (99)$$

where  $\Lambda_t$  is some non-decreasing process which is constant on any subinterval of  $\{t : X_t > 0\}$ . In this section we will show that  $\Lambda_t \equiv 0$ .

Remember that  $\Sigma = \{t : X_t = 0\}$ . The *index* of a Bessel process is defined as

$$\nu = \frac{\delta}{2} - 1. \quad (100)$$

Notice that  $\nu = -1/4$  in our case, Bessel process of dimension  $3/2$ .

**Lemma 5.7.**  $\dim_{\mathcal{H}}(\Sigma) = 1/4$

*Proof.* For fixed  $\varepsilon > 0$  choose  $\varepsilon_1 > 0$  such that

$$\frac{1-\varepsilon}{3} \frac{1}{m^2} \leq \frac{1}{3} \frac{(3m^4 + 2m^2 + 1)(m^2 - 1)^2}{m^2(m^2 + 1)^2} \leq \frac{1}{3} \frac{1}{m^2} \quad (101)$$

$$\frac{1}{2(1+\varepsilon)^{1/2}} \sqrt{x} \leq \sqrt{1 + \frac{1}{x}} - \sqrt{\frac{1}{x}} \leq \frac{1}{2} \sqrt{x} \quad (102)$$

for all  $0 < m \leq \sqrt{\varepsilon_1}/2$  and  $0 < x \leq \varepsilon_1$ . Then whenever  $t$  is such that  $X_t/Y_t \leq \varepsilon_1$ , it is possible to estimate

$$\frac{4(1-\varepsilon)}{3} \frac{1}{X_t} \leq (\text{drift of } X_t) \leq \frac{4(1+\varepsilon)}{3} \frac{1}{X_t} \quad (103)$$

and hence we can couple  $(X_t)_{t \in [\tau, \tilde{\tau}]}$ , where  $\tau$  is any stopping time such that  $X_\tau \leq \varepsilon_1/2$  and  $\tilde{\tau} = \inf\{t \geq \tau : X_t = \varepsilon_1\}$ , to (scaled) Bessel processes  $X'_t$  and  $X''_t$  which satisfy

$$dX'_t = \frac{4}{\sqrt{3}} dB_t + \frac{4(1-\varepsilon)}{3} \frac{dt}{X'_t} \quad (104)$$

$$dX''_t = \frac{4}{\sqrt{3}} dB_t + \frac{4(1+\varepsilon)}{3} \frac{dt}{X''_t} \quad (105)$$

and  $X_\tau = X'_\tau = X''_\tau$ . Under this coupling  $X'_t \leq X_t \leq X''_t$  for any  $t \in [\tau, \tilde{\tau}]$ .

Clearly

$$\{t : X''_t = 0\} \subset \{t : X_t = 0\} \subset \{t : X'_t = 0\}. \quad (106)$$

By [14] Exercise XI.1.25 and [3] Theorem III.15, the Hausdorff dimension of the support of the local time of a Bessel process with index  $\nu \in [-1, 0]$  is  $-\nu$ . By Markov property of the Bessel process, the support of the local time is the entire set of times when the process is at the origin. Hence the set  $\Sigma$  is sandwiched between two sets which have dimension arbitrarily close to  $1/4$ .  $\square$

Now  $\Lambda_t \equiv 0$  follows from Lemma 5.6 and from the fact that  $(U_t)_{t \in [0, T]}$  is  $1/2 - \varepsilon$  Hölder continuous for any  $\varepsilon > 0$ .

**5.5.4. The process as a SLE $[\kappa, Z]$  process.** Generally the drift of a SLE can be given in terms of a partition function  $Z(u, v, \dots)$ . We call such process SLE $[\kappa, Z]$  and the driving process  $U_t$  is given by

$$dU_t = \sqrt{\kappa} dB_t + \kappa \partial_u \log Z(U_t, V_t, \dots) dt. \quad (107)$$

The other points follow the Loewner equation. The partition function is not unique, but if we require Möbius covariance and finite limit as  $w \rightarrow \infty$ , then

$$Z(u, v, w) = \frac{1}{y^{1/8} (m^2 + 1)^{1/4} m^{1/4}} \quad (108)$$

where  $y = w - v$ ,  $m = \sqrt{1 + y/x} - \sqrt{y/x}$  and  $x = v - u$ .

It is possible to use this result to show that the interface when conditioned on an internal arc pattern  $(a \frown d, b \frown c)$  converges towards so called hypergeometric SLE [13].

## 6. PROOF OF THEOREM 1.1

As a conclusion to this entire article we outline below the proof of Theorem 1.1.

The basic crossing estimates were established for the FK Ising exploration tree and its branches in Section 3.2, Theorem 3.4. Based on those estimates, the pre-compactness of probability laws of a single branch or a finite subtree (i.e., a subtree with a fixed number of target points) was shown in Section 3.7. By those results we

can choose convergent subsequences. The structure of the tree is characterized by the target independence, the independence of the branches after disconnection and the martingale characterization of a single branch in Section 5. Here it is needed that the branches converge in the strong sense as capacity parameterized curves.

By these results, every sequence of finite subtrees of the approximating domains converges in distribution to a finite subtree of the  $\text{SLE}(\kappa, \kappa - 6)$  exploration tree with  $\kappa = 16/3$ . The convergence takes place, for instance, in the unit disc  $\mathbb{D}$  after a conformal transformation and under the metric defined in Section 3.3.1. In fact, it is possible to extend this convergence to the original domain (without the conformal transformation). See Corollary 1.8 in [11] for such a result.

The precompactness of the probability laws of the full tree and of the loop collection were established in Theorems 3.1 and 3.7, respectively. Therefore we can choose subsequences such that both the full tree and the loop collection converge. The above together with the finite-tree approximation in Theorem 3.6 implies that the full tree has a unique limit which is the  $\text{SLE}(\kappa, \kappa - 6)$  exploration tree with  $\kappa = 16/3$ . Similarly using the finite-tree approximation in Theorem 3.8 we show that the loop collection has a unique limit which is characterized by the one-to-one correspondence to the exploration tree under the maps introduced in Sections 1.2.1 and 2.3, see Theorem 3.10. This ends the outline of the proof.

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#### APPENDIX A. DISTORTION OF ANNULI UNDER CONFORMAL MAPS

Let  $\Omega$  be a simply connected domain and  $\phi : \mathbb{D} \rightarrow \Omega$  is a conformal map. The next two lemmas establish bounds for distortion of annuli in terms of their conformal modulus which is a constant times  $\log(R/r)$  for an annulus  $A = A(z_0, r, R)$ . The proofs can be found from Section 6.3.3 of [10].

**Lemma A.1** (Distortion of annuli contained in  $\mathbb{D}$ ). *For any  $\rho' > 1$  sufficiently large, there exists  $\rho > 1$  such that the following holds. Suppose that the annulus  $A = A(z_0, r, R) \subset \mathbb{D}$  is such that  $R/r > \rho$  and  $R \leq \frac{1}{2}(1 - |z_0|)$ . Then there exists an annulus  $A' = A(z'_0, r', R')$  with  $R'/r' > \rho'$  such that  $A' \subset \phi(A)$  and  $A'$  separates the components of  $\phi(A)$  in  $\mathbb{C}$ . Furthermore the dependency of  $\rho$  and  $\rho'$  can be made linear.*

**Lemma A.2** (Distortion of annuli not fully contained in  $\mathbb{D}$ ). *For any  $\rho' > 1$  sufficiently large, there exists  $\rho > 1$  such that the following holds. Suppose that the annulus  $A = A(z_0, r, R)$  is such that  $R/r > \rho$ ,  $1 - |z_0| < r$  and  $R < 1$  (that is,  $\partial\mathbb{D}$  crosses  $A$ ). Then there exists an annulus  $A' = A(z'_0, r', R')$  with  $R'/r' > \rho'$  such that there exists a connected component  $O$  of  $\Omega \cap A'$  such that  $O \subset \phi(A \cap \mathbb{D})$  and  $O$  separates the components of  $\phi((\partial A) \cap \mathbb{D})$  in  $\Omega$ .*

#### APPENDIX B. AUXILIARY RESULTS ON A PRIORI BOUNDS

Denote conformal images of the FK Ising boundary loop exploration tree and the random cluster configuration under  $\phi^{-1} : \Omega \rightarrow \mathbb{D}$  by  $\mathcal{T}_{\mathbb{D}}$  and  $\omega_{\mathbb{D}}$ , respectively. The following auxiliary results on the relation of the number of crossings by disjoint

segments of tree branches and the number of disjoint open or dual-open crossings could be written in the original domain or any other “reference” domain. The unit disc is nice in the sense that the annulus has two sides on the boundary and thus the number  $n - 1$  appears in the formulation of the second lemma instead of a smaller number needed when the boundary makes more crossings of the annulus.

**Lemma B.1.** *Suppose that  $A = A(z_0, r, R) \subset \mathbb{D}$ . If there are at least  $2n$  disjoint segments of the tree  $\mathcal{T}_{\mathbb{D}}$  crossing  $A$ , then there are  $n$  dual-open arms in the random cluster configuration  $\omega_{\mathbb{D}}$ .*

*Proof.* Any crossing  $A$  consists only of part of a branch that follows a single loop. Thus its left-hand side is entirely open and the right-hand side entirely dual open. If there are exactly  $m = 2n$  crossings of  $A$ , then due to topological reasons the adjacent minimal crossings have both be open or dual open. Thus in every second pair of adjacent minimal crossings have to have a dual-open crossing between them.  $\square$

**Lemma B.2.** *Suppose that  $A = A(z_0, r, R)$  such that  $B(z_0, r) \cap \partial\mathbb{D} \neq \emptyset$ . If there are at least  $2n$  disjoint segments of the tree  $\mathcal{T}_{\mathbb{D}}$  crossing  $A$ , then there are  $n - 1$  dual-open arms in the random cluster configuration  $\omega_{\mathbb{D}}$ .*

*Proof.* Suppose that there are exactly  $k$  minimal crossings of  $A$  which touch the boundary so that their dual-open right-hand side touches free boundary (i.e. the vertex sets of the dual-open path and the dual boundary intersect) and that along each them there is at least one branching point. Then due to topological reasons  $k$  is at most 2. Suppose also that there are  $m$  crossings in total, then  $m - k$  crossings disjoint from the boundary or touch the boundary by their open left-hand sides. In particular, for these types of crossings, the crossings consists only of a part following a single loop. Thus the entire right-hand side is dual-open. Similarly as in the previous proof, we see that there are at least  $\lfloor (m - k)/2 \rfloor$  dual-open crossings of  $A$ .  $\square$

## APPENDIX C. AUXILIARY RESULTS ON LOEWNER EVOLUTIONS

For a Loewner chain  $(K_t)_{t \in [0, T]}$  in  $\mathbb{H}$  driven by  $(U_t)_{t \in [0, T]}$ , let  $(V_t)_{t \in [0, T]}$  be the function defined by

$$V_t = g_t(\sup(K_t \cap \mathbb{R})).$$

The point  $V_t$  is thus the image of the rightmost point on the hull under the conformal map  $g_t$ . The following lemma can be extracted from the proof of Proposition 3.12 in [17].

**Lemma C.1.** *For any  $(U_t)_{t \in [0, T]}$  and  $(V_t)_{t \in [0, T]}$  as above, it holds that  $\int_0^T \mathbb{1}_{V_t = U_t} dt = 0$ .*

**Proposition C.2.** *Suppose that  $\int_0^T \frac{dt}{V_t - U_t} < \infty$ . There exists a non-decreasing function  $(\Lambda_t)_{t \in [0, T]}$  with  $\Lambda_0 = 0$  such that*

$$V_t = V_0 + \int_0^t \frac{2ds}{V_s - U_s} + \Lambda_t. \quad (109)$$

*Proof.* Let  $\varepsilon > 0$ ,  $J_t^\varepsilon = \min\{k\varepsilon > \sup(K_t \cap \mathbb{R}) : k \in \mathbb{Z}\}$  and  $\tilde{V}_t^\varepsilon = g_t(J_t^\varepsilon)$ . Then it follows from monotonicity of  $g_t$  that  $V_t \leq V_t^\varepsilon \leq V_t + \varepsilon$ . By the Loewner equation we can write

$$\tilde{V}_t^\varepsilon = \tilde{V}_0^\varepsilon + \int_0^t \frac{2ds}{\tilde{V}_s^\varepsilon - U_s} + \sum_{s \leq t} \xi_s^\varepsilon \quad (110)$$



where  $\xi_s^\varepsilon \in [0, \varepsilon]$  and  $\xi_s^\varepsilon \neq 0$  only when  $\tilde{V}_u^\varepsilon - U_u$  hits zero as  $u \nearrow s$ . Thus the sum on the right-hand side is a finite sum. By Lemma C.1 and Lebesgue's dominated convergence theorem the middle term in (110) converges to the integral in (109). Since also  $\tilde{V}_t^\varepsilon$  converges uniformly to  $V_t$ , it holds that  $\sum_{s \leq t} \xi_s^\varepsilon$  converges uniformly to some continuous function  $\Lambda_t$  as  $\varepsilon \rightarrow 0$ . It follows that  $\Lambda_t$  is non-decreasing and  $\Lambda_0 = 0$ . The claim follows.  $\square$

*Remark C.3.* Notice that in fact,  $\int_0^T \frac{dt}{\tilde{V}_t^\varepsilon - U_t} < \infty$  always. Namely,

$$\int_0^T \frac{dt}{\tilde{V}_t^\varepsilon - U_t + \varepsilon} \leq \int_0^T \frac{dt}{\tilde{V}_t^\varepsilon - U_t} \leq \frac{1}{2} (\tilde{V}_T^\varepsilon - \tilde{V}_0^\varepsilon) \leq \frac{1}{2} (V_T - V_0 + \varepsilon).$$

Thus  $\int_0^T \frac{dt}{\tilde{V}_t^\varepsilon - U_t} < \infty$  follows from Lebesgue's monotone convergence theorem.

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